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# Robust optimal investment–reinsurance strategies for an insurer with multiple dependent risks ☆

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A B S T R A C T

This paper considers a robust optimal investment and reinsurance problem with multiple dependent risks for an Ambiguity-Averse Insurer (AAI), who is uncertain about the model parameters. We assume that the surplus of the insurance company can be allocated to the financial market consisting of one risk-free asset and one risky asset whose price process satisfies square root factor process. Under the objective of maximizing the expected utility of the terminal surplus, by adopting the technique of stochastic control, closed-form expressions of the robust optimal strategy and the corresponding value function are derived. The verification theorem is also provided. Finally, by presenting some numerical examples, the impact of some parameters on the optimal strategy is illustrated and some economic explanations are also given. We find that the robust optimal reinsurance strategies under the generalized mean–variance premium are very different from that under the variance premium principle. In addition, ignoring model uncertainty risk will lead to significant utility loss for the AAI.



## 1. Introduction

The insurance market plays a quite important role in the global financial market. The stable operation of insurers influences the stability of the financial market. Reinsurance and investment are two significant issues for the insurers. On one hand, through the reinsurance, the large claims risk of the insurer is transferred to other insurers, so it is an effective risk-spreading approach. On the other hand, by investing the surplus into the financial market, the insurer can increase her wealth and enhance the market competitiveness. Recently, based on different decision-making objectives such as minimizing the ruin probability of the insurer, maximizing the expected utility of terminal surplus or mean–variance criteria, much attention has been paid to the research on the optimal reinsurance or/and investment problem. For example, the related literature can refer to Cao and Wan (2009), Liang and Guo (2011), Zhang et al. (2016a,b), and Xu et al. (2017) etc.

In the afore-mentioned literature, the insurance company is assumed to have only one business. In fact, many insurance companies have two or more lines of business, and most of them are not independent of each other due to the risk of suffering from a common claim shock. For example, the auto insurance and third party insurance, the casualty insurance and health insurance are all dependent lines of business. Thus, many scholars begin to investigate the optimal investment or reinsurance strategy under the multivariable dependent risks. For example, Bai et al. (2013) firstly convert the two-dimensional compound Poisson reserve risk process into a two-dimensional diffusion approximation process, and derive the optimal reinsurance strategy to minimize the ruin probability of the insurer. Under the variance premium principle and the objective of maximizing the expected exponential utility of terminal surplus, Liang and Yuen (2016) obtain the optimal proportional reinsurance strategy when the surplus of insurance company is described by a two-dimensional dependent compound Poisson process and its diffusion approximation, respectively. Meanwhile, Yuen et al. (2015) extend the work of Liang and Yuen (2016) to the risk model with multiple dependent classes of insurance business. Ming et al. (2016) investigate the optimal reinsurance strategy with common shock dependence based on mean–variance criteria. Later, Bi et al. (2016) extend the model of Ming et al. (2016) to the case that the surplus

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can be invested in the financial market, and both the optimal investment and reinsurance strategies are obtained.

In the traditional investment or reinsurance model, the parameters are assumed to be fixed constant or deterministic function of time, i.e., the insurer has no doubt and is quite sure about the accuracy of the estimated parameters. However, in practice, it is difficult to estimate the parameters of the model with precision, especially the expected return of the risky asset. The same uncertainty exists in the surplus process of the insurance company, which results in the so-called model uncertainty. Usually, the model, with the parameters of which are estimated under the real probability measure $\mathbb{P}$, is called the reference model relative to the real but unknown model. Due to the inevitability of statistical error, the reference model deviates from the real model more or less. According to the existing literature, the optimal reinsurance or investment strategy depends on the model parameters. The non robustness of the estimated parameters will lead the optimal strategy to be unstable. When the insurer is aware of the risk of model uncertainty, she will take model uncertainty into consideration during the decision-making process. In this case, the insurer is called ambiguity-averse and she prefers the optimal strategy which is robust to model misspecification. Thus, model uncertainty, which has remarkable impact on the optimal investment and reinsurance strategies, has to be taken into consideration in the process of reinsurance arrangement and asset allocation.

Currently, the main approach of dealing with model uncertainty is the robust control approach developed by Anderson et al. (1999). The fundamental idea behind this method lies in that the decision-maker takes the reference model as a starting point, and she knows that the reference model cannot describe the real insurance or financial market correctly. Therefore an alternative model needs to be incorporated. The decision-maker tries to find the optimal strategy among the family of alternative models which does not deviate from the reference model too much, and she can find the optimal robust strategy which is the best choice in the worst case model. Based on the above approach, Yi et al. (2013) consider a robust optimal reinsurance and investment problem under Heston's stochastic volatility model for an insurer with ambiguity aversion, the closed-form expression of the optimal strategy is obtained under the objective of maximizing the expected exponential utility. Pun and Wong (2015) investigate the robust investment–reinsurance problem with more generally multiscale stochastic volatility. When the price process of the risky asset satisfies constant elasticity of variance (CEV) model, Zheng et al. (2016) derive the robust optimal investment and proportional reinsurance strategies. To investigate the influence of the misspecification for jump parameter on the optimal strategy of the insurer, Li et al. (2018) consider the robust optimal excess-of-loss reinsurance and investment strategies for the model with jumps. Under the mean–variance criteria, Yi et al. (2015) obtain the robust optimal reinsurance and investment strategies with a benchmark. Zeng et al. (2016) derive the robust equilibrium reinsurance–investment strategies with jumps in the framework of game theory.

In the literature mentioned above, the robust optimal reinsurance or/and investment problems are investigated under the risk model with only one business for an insurer. To the best of our knowledge, there is little research on the robust optimal decision-making problem under the multiple dependent risks for an insurer. In this paper, we focus on the effect of uncertainty about the diffusion risk arising from risky asset and surplus process of the insurer, and consider the robust optimal investment and reinsurance problem with multiple dependent risks. We assume that the insurer adopts proportional reinsurance to disperse risk. Refer to Zhang et al. (2016a), the reinsurance premium is calculated under the generalized mean–variance premium principle, which includes the expected value principle and the variance principle as special cases. The surplus of the insurance company can be invested in the financial market consisting of one risk-free asset and a risky asset or a market index. Inspired by Shen and Zeng (2015) and Li et al. (2017), the price process of the risky asset is assumed to satisfy a square-root factor process, which can describe the randomness of volatility. We assume that the insurer is both risk and ambiguity averse. Thus, under the objective of maximizing expected exponential utility, using the method of robust optimal control, we obtain the closed-form expressions of optimal investment and reinsurance strategies and corresponding value function. To evaluate the utility loss which is caused by neglecting the risk of model uncertainty in the decision-making process, we investigate a suboptimal strategy and define a wealth-equivalent utility loss function. Finally, some numerical analyses are given to present the sensitivity of the optimal strategy and utility loss value on some parameters.

The main contribution of this paper is listed as follows. (i) The robust optimal investment–reinsurance strategies under the multiple dependent risks are investigated firstly. (ii) The reinsurance premium of the multiple dependent risks is calculated based on the generalized mean–variance premium principle. (iii) The price process of the risky asset satisfies affine-form square-root factor model, which is an even generalized model and makes the CEV or Heston's stochastic volatility model as special cases. So our paper extends some existing models. In our paper, for the optimal control problem under the diffusion risk model with multiple dependent classes of business, the reinsurance proportion for every class of business is required to belong to $[0, 1]$. The present constraints make the mathematical solution for the robust optimal reinsurance strategies become more complex. Comparing with the results from the existing literature we find some novel results: (i) the robust optimal reinsurance strategies under the generalized mean–variance premium principle depend on not only the safety loading, time and interest rate, but also the ambiguity-aversion coefficient, the claim amount and intensity parameters. This is different from the optimal reinsurance strategies under the variance premium principle (Liang and Yuen, 2016). They show that the optimal reinsurance strategies only depend on the safety loading, time and interest rate. (ii) Yuen et al. (2015) and Liang and Yuen (2016) only investigated the optimal proportional reinsurance strategies. In our model, both robust optimal investment and reinsurance strategies are obtained. We find that the robust optimal investment strategy only depends on the interest rate, ambiguity-aversion coefficient, and the financial market parameters, which is independent of the parameters in insurance market.

The rest of this paper is organized as follows. Assumption and problem formulation are described in Section 2. In Section 3, the robust optimization problem is solved, the closed-form of the optimal investment and reinsurance strategies are derived. In Section 4, a suboptimal strategy is considered and a utility loss function is defined. Section 5 presents some numerical illustration to analyze our theoretical results and investigates the sensitivity of optimal strategy on some parameters. Section 6 concludes our work.

## 2. Assumption and problem formulation

In this section, we will give some assumptions for the insurance and financial market. We suppose that the risky asset can be traded continuously over time, and no transaction costs or taxes are considered. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the usual conditions, i.e., $\{\mathcal{F}_t\}_{t\in[0,T]}$ is right-continuous and $\mathbb{P}$-complete, where $T$ is a positive finite constant representing the investment time horizon of an insurance company.

### 2.1. The surplus process

In this paper, we suppose that an insurance company has multiple dependent lines of business such as auto insurance, third party insurance, casualty insurance, health insurance and so on. Let $z(\geq 2)$ be the number of the dependent business which are managed by an insurance company, $X_i^{(l)}(i = 1, 2, \ldots)$ are the claim size random variables for the $l$th $(l = 1, 2, \ldots, z)$ line of business with common distribution $F_l(x)$. The first-order and second-order moments of the variable $X_i^{(l)}$ are denoted by $\mu_l = \mathbb{E}[X_i^{(l)}]$, and $\nu_l = \mathbb{E}[(X_i^{(l)})^2]$. Let $N_1(t), N_2(t), \ldots, N_z(t)$ and $N(t)$ be $z+1$ independent Poisson processes with intensity parameters $\lambda_1, \lambda_2, \ldots, \lambda_z$ and $\lambda$, respectively. Denote $X(t)$ as the total surplus of the insurance company up to time $t$, thus,

$$X(t) = x_0 + ct - \sum_{l=1}^{z} S_l(t), \quad t \geq 0, \tag{1}$$

where $x_0$ is initial surplus. For $l = 1, 2, \ldots, z$, $N_l(t) + N(t)$ represents the total claim number for the $l$th classes of business at time interval $[0, t]$, and $S_l(t) = \sum_{i=1}^{N_l(t)+N(t)} X_i^{(l)}$ is the aggregate claims process generated from the class $l$. It is obvious that the $z$ classes of business subject to a common shock are governed by the counting process $N(t)$. Thus, the claims process among the $z$ classes of business is related to each other. $c$ is the rate of premium, which is calculated according to the expected value premium principle with positive safety loading $\chi$, namely,

$$c = (1+\chi)\Big(\sum_{l=1}^{z}(\lambda_l + \lambda)\mu_l\Big). \tag{2}$$

To be protected from potential large claims, at each moment, the insurer is allowed to purchase proportional reinsurance to disperse risk, and let $q_l(t) \in [0, 1]$ be the reinsurance retention levels for the $l$th $(l = 1, 2, \ldots, z)$ line of business at time $t$. i.e., for a claim amount $X_i^{(l)}$ occurring at time $t$, the insurer pays the claim amount $q_l(t)X_i^{(l)}$, while the reinsurer pays $(1-q_l(t))X_i^{(l)}$. Let $X^q(t)$ be the surplus process of the insurer associated with the strategy $q(t) = (q_1(t), q_2(t), \ldots, q_z(t))$. The dynamics of $X^q(t)$ can be described by

$$\mathrm{d}X^q(t) = [c - \delta(q(t))]\mathrm{d}t - \sum_{l=1}^{z} q_l(t)\mathrm{d}S_l(t), \tag{3}$$

where $\delta(q(t))$ is the reinsurance premium rate. From Grandell (1991), we know that the compound Poisson processes $S_l(t)$ can be approximated by the following diffusion process $\hat{S}_l(t)$:

$$\mathrm{d}\hat{S}_l(t) = a_l \mathrm{d}t + \sigma_l \mathrm{d}B_l(t), l = 1, 2, \ldots, z, \tag{4}$$

where $a_l = (\lambda_l + \lambda)\mu_l$, $\sigma_l^2 = (\lambda_l + \lambda)\nu_l$. Here $B_i(t)$ and $B_j(t)$ for $\forall i \neq j, i, j = 1, 2, \ldots, z$ are standard Brownian motions with correlation coefficient $\rho_{ij} \in (-1, 1)$. Then by Bai et al. (2013) and Liang and Yuen (2016), the diffusion approximation of the surplus process evolves as

$$\mathrm{d}X^q(t) = [c - \delta(q(t)) - \sum_{l=1}^{z} a_l q_l(t)]\mathrm{d}t + \sqrt{\sum_{l=1}^{z} \sigma_l^2 q_l^2(t) + \sum_{i \neq j}^{z} q_i(t)q_j(t)\lambda\mu_i\mu_j}\,\mathrm{d}W_0(t), \tag{5}$$

where $W_0(t)$ is a standard Brownian motion.

In this paper, we adopt the generalized mean–variance premium principle (see Zhang et al. (2016a)) to calculate the reinsurance premium, which includes the expected value principle and variance principle as special cases. i.e., with the nonnegative safety loading $\eta$ and $\xi$, the reinsurance premium is $(1+\eta)[\mathbb{E}(\cdot) + \xi\mathbb{V}\mathrm{ar}(\cdot)]$. So

$$\delta(q(t)) = (1+\eta)\Big[\sum_{l=1}^{z}(1 - q_l(t))a_l + \xi h(q(t))\Big], \tag{6}$$

where $h(q(t)) = \sum_{l=1}^{z}(1 - q_l(t))^2\sigma_l^2 + \sum_{i \neq j}^{z}(1 - q_i(t))(1 - q_j(t))\lambda\mu_i\mu_j$.

**Remark 2.1.** When $\eta > 0, \xi = 0$, the reinsurance premium rate degenerates to the expectation premium rate.

**Remark 2.2.** When $\eta = 0, \xi > 0$, the reinsurance premium rate is calculated under the variance premium principle.

### 2.2. The financial market

We assume that the self-financing insurer is allowed to invest the surplus into the financial market consisting of one risk-free asset and one risky asset. The price process $B(t)$ of the risk-free asset satisfies

$$\mathrm{d}B(t) = rB(t)\mathrm{d}t, \tag{7}$$

where $r(> 0)$ represents the risk-free interest rate. The price process $S(t)$ of the risky asset is described by the following stochastic volatility model, i.e.,

$$\mathrm{d}S(t) = S(t)[\mu(t)\mathrm{d}t + \sigma(t)\mathrm{d}W_1(t)], \tag{8}$$

where $\mu(t)$ is the appreciation rate and $\sigma(t)$ is the volatility rate, $W_1(t)$ is a standard Brownian motion. We define $\varphi(t) = \frac{\mu(t)-r}{\sigma(t)}$ for $\forall t \in [0, T]$ as the market price process of the risk. Referring to Shen and Zeng (2015) and Li et al. (2017), we suppose that the process $\{\varphi(t)\}_{t\in[0,T]}$ is related to a stochastic factor process $\{\alpha(t)\}_{t\in[0,T]}$, i.e.,

$$\varphi(t) = \theta\sqrt{\alpha(t)}, \quad \forall t \in [0, T], \quad \theta \in \mathbb{R}_0 : \mathbb{R}\backslash\{0\}, \tag{9}$$

where the stochastic factor process $\{\alpha(t)\}_{t\in[0,T]}$ satisfies the following affine-form mean-reverting square root model

$$\mathrm{d}\alpha(t) = k[\phi - \alpha(t)]\mathrm{d}t + \sqrt{\alpha(t)}[k_1\mathrm{d}W_1(t) + k_2\mathrm{d}W_2(t)], \quad \alpha(0) = \alpha_0 \geq 0, \tag{10}$$

where $k, \phi, k_1, k_2$ are all positive constants and $W_2(t)$ is another standard Brownian motion. In addition, we assume that the above Brownian motions $W_0(t)$, $W_1(t)$ and $W_2(t)$ are mutually independent.

**Remark 2.3.** In model (8), for the appreciation rate $\mu(t)$ and the volatility rate $\sigma(t)$ of the risky asset, we assume that at least one of them is a stochastic process and simultaneously related to the stochastic factor process $\alpha(t)$ satisfying $\frac{\mu(t)-r}{\sigma(t)} = \theta\sqrt{\alpha(t)}$. The reason for this setting is to make the following wealth process (11) have an unique state variable, which makes the optimal control problem of this paper tractable. Of course, under this assumption, the market price of risk, $\varphi(t)$, still keeps as a stochastic process related to mean-reverting square root process $\alpha(t)$. It is worth pointing out that the optimal control problem with weaker assumption between the drift and volatility is investigated in Pun and Wong (2015). In their model, the risky asset is assumed to follow a multiscale stochastic volatility (SV) model, and an investment–reinsurance strategy that well approximates the optimal strategy of the robust optimization problem under the multiscale SV model is derived.

**Remark 2.4.** If $\alpha(t) = S^{-2\beta}(t)$, $\mu(t) = \mu$, $\sigma(t) = \varrho/\sqrt{\alpha(t)} = \varrho S^{\beta}(t)$, $k = 2\beta\mu$, $\phi = (\frac{1}{2}+\beta)\frac{\varrho^2}{\mu}$, $k_1 = -2\beta\varrho$, $k_2 = 0$, $\theta = \frac{\mu-r}{\varrho}$, where $\mu, \varrho > 0$. Then the risky asset's price process degenerates into CEV model. Here $\beta$ is the elasticity parameter. When $\beta < 0$, the instantaneous volatility $\sigma(t)$ will increase as the stock price decreases. When $\beta > 0$, the situation is reversed. In reality, many authors have examined that $\beta < 0$ is more realistic.

**Remark 2.5.** If $\mu(t) = r + \theta\alpha(t)$, $\sigma(t) = \sqrt{\alpha(t)}$, $k_1 = \sigma_0\rho$, $k_2 = \sigma_0\sqrt{1-\rho^2}$, where $\theta \in R_0$, $\sigma_0 > 0$, $\rho \in (-1, 1)$, moreover, if the Feller condition $2k\phi \geq \sigma_0^2$ is satisfied to guarantee $\alpha(t) > 0$, then the price process of the risky asset degenerates to Heston's stochastic volatility model.

Denote $\pi(t)$ as the total amount of the insurer's surplus invested in the risky asset at time $t$, and the rest of the surplus is invested in the risk-free asset. Then we define the decision making process of the insurer as $u(t) = \{(\pi(t), q(t)), t \in [0, T]\}$. Thus, the surplus process associated with strategy $u(t)$ is given by

$$\begin{aligned}
\mathrm{d}X^u(t) &= [c - \delta(q(t)) - \sum_{l=1}^{z} a_l q_l(t)]\mathrm{d}t \\
&\quad + \sqrt{\sum_{l=1}^{z}\sigma_l^2 q_l^2(t) + \sum_{i\neq j}^{z} q_i(t)q_j(t)\lambda\mu_i\mu_j}\,\mathrm{d}W_0(t) \\
&\quad + \pi(t)\frac{\mathrm{d}S(t)}{S(t)} + [X^u(t) - \pi(t)]\frac{\mathrm{d}B(t)}{B(t)} \\
&= [c - \delta(q(t)) - \sum_{l=1}^{z} a_l q_l(t) + \pi(t)(\mu(t) - r) + rX^u(t)]\mathrm{d}t \\
&\quad + \sqrt{\sum_{l=1}^{z}\sigma_l^2 q_l^2(t) + \sum_{i\neq j}^{z} q_i(t)q_j(t)\lambda\mu_i\mu_j}\,\mathrm{d}W_0(t) \\
&\quad + \pi(t)\frac{\mu(t) - r}{\theta\sqrt{\alpha(t)}}\mathrm{d}W_1(t).
\end{aligned} \tag{11}$$

**Remark 2.6.** From Eq. (11), we find that the surplus process $X^u(t)$ depends on $\mu(t)$ and $\alpha(t)$. Recall Remark 2.3, we know that if the process $\mu(t)$ is stochastic, which is assumed to be the function of $\alpha(t)$. Thus, in the surplus process (11), we see $\{\alpha(t)\}_{t\in[0,T]}$ as the unique state process.

### 2.3. Robust optimal control problem for an AAI

Now we suppose that the insurer is interested in maximizing the expected utility of the surplus at the terminal time $T$, and the insurer is assumed to have exponential utility $U(x) = -\frac{1}{m}\exp(-mx)$, where $m > 0$ is a constant representing the absolute risk aversion coefficient. In traditional model, the insurer is assumed to be an ambiguity-neutral investor (ANI) with objective function as

$$\sup_{u\in\tilde{\mathcal{U}}}\mathbb{E}^{\mathbb{P}}[U(X^u(T))] = \sup_{u\in\tilde{\mathcal{U}}}\mathbb{E}^{\mathbb{P}}[-\frac{1}{m}e^{-mX^u(T)}], \tag{12}$$

where $\tilde{\mathcal{U}}$ is the set of admissible strategy $u$ in a given market, and $\mathbb{E}^{\mathbb{P}}$ is the expectation under the real probability measure $\mathbb{P}$. Under the traditional model, the insurer is usually assumed to be risk averse, and she has no doubt about the model parameters. In this case, we call the insurer as to be ambiguous-neutral. However, during the asset allocation process of the surplus wealth, it is hard to get the real and precise estimation for the related parameters such as the expected return rate of risky asset, drift rate of the stochastic factor process and the approximated diffusion process of the surplus. The model, which is established by statistics method under the real-world data and probability measure $\mathbb{P}$, can only be taken as the reference model to the real. Correspondingly, the parameters are called the reference model parameters. Due to the estimation error, the insurer is uncertain about the reference model. He is aware that he does not know the real model, but only looks at the reference model as an approximation of the real model. Since the decision-making process depends on the value of parameters, the uncertainty of parameters' estimation will directly affect the robustness of the strategy. To obtain the robust investment and reinsurance strategies, one way is to reconsider the model and parameters under an alternative measure which is equivalent to the real-world measure. Now we try to construct the set of alternative measures which are equivalent to $\mathbb{P}$, i.e., $\mathcal{Q} := \{\mathbb{Q}|\mathbb{Q} \sim \mathbb{P}\}$. According to Girsanov's theorem, for each $\mathbb{Q} \in \mathcal{Q}$, there exists a progressively measurable process $\gamma(t) = (\gamma_0(t), \gamma_1(t), \gamma_2(t))^{\top}$, such that $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \zeta(T)$, where

$$\zeta(t) = \exp\left\{-\int_0^t \gamma(s)^{\top}\mathrm{d}W(s) - \frac{1}{2}\int_0^t \|\gamma(s)\|^2\mathrm{d}s\right\} \tag{13}$$

with $W(t) = (W_0(t), W_1(t), W_2(t))^{\top}$, and $\|\gamma(t)\|^2 = \gamma_0^2(t) + \gamma_1^2(t) + \gamma_2^2(t)$. If $\gamma(t)$ satisfies Novikov's condition (we will give the technical condition in Remark 3.1)

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left\{\int_0^T \frac{1}{2}\|\gamma(s)\|^2\mathrm{d}s\right\}\right] < \infty,$$

then referring Karatzas and Shreve (2012), $\zeta(t)$ is a $\mathbb{P}-$martingale with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. Further more, by Girsanov's theorem the Brownian motions under $\mathbb{Q} \in \mathcal{Q}$ can be defined as

$$\mathrm{d}W_0^{\mathbb{Q}}(t) = \gamma_0(t)\mathrm{d}t + \mathrm{d}W_0(t),$$

$$\mathrm{d}W_1^{\mathbb{Q}}(t) = \gamma_1(t)\mathrm{d}t + \mathrm{d}W_1(t),$$

$$\mathrm{d}W_2^{\mathbb{Q}}(t) = \gamma_2(t)\mathrm{d}t + \mathrm{d}W_2(t).$$

So under the alternative measure $\mathbb{Q}$, the dynamic price process $S(t)$ of the risky asset, the stochastic factor process $\alpha(t)$ and the surplus process $X^u(t)$ can be rewritten as

$$\mathrm{d}S(t) = S(t)[(\mu(t) - \gamma_1(t)\sigma(t))\mathrm{d}t + \sigma(t)\mathrm{d}W_1^{\mathbb{Q}}(t)], \tag{14}$$

$$\begin{aligned}
\mathrm{d}\alpha(t) &= [k(\phi - \alpha(t)) - \sqrt{\alpha(t)}(k_1\gamma_1(t) + k_2\gamma_2(t))]\mathrm{d}t \\
&\quad + \sqrt{\alpha(t)}[k_1\mathrm{d}W_1^{\mathbb{Q}}(t) + k_2\mathrm{d}W_2^{\mathbb{Q}}(t)],
\end{aligned} \tag{15}$$

$$\begin{aligned}
\mathrm{d}X^u(t) &= \Big\{c - \delta(q(t)) - \sum_{l=1}^{z} a_l q_l(t) - \gamma_0(t) \\
&\quad \times \sqrt{\sum_{l=1}^{z}\sigma_l^2 q_l^2(t) + \sum_{i\neq j}^{z} q_i(t)q_j(t)\lambda\mu_i\mu_j} \\
&\quad + \pi(t)[\mu(t) - r - \gamma_1(t)\frac{\mu(t) - r}{\theta\sqrt{\alpha(t)}}] + rX^u(t)\Big\}\mathrm{d}t \\
&\quad + \sqrt{\sum_{l=1}^{z}\sigma_l^2 q_l^2(t) + \sum_{i\neq j}^{z} q_i(t)q_j(t)\lambda\mu_i\mu_j}\,\mathrm{d}W_0^{\mathbb{Q}}(t) \\
&\quad + \pi(t)\frac{\mu(t) - r}{\theta\sqrt{\alpha(t)}}\mathrm{d}W_1^{\mathbb{Q}}(t).
\end{aligned} \tag{16}$$

Now we give the definition of the set of admissible strategies under the expected utility objective. Due to the fact that the proof of Theorem 3.1 in the next section needs to use the result from Kraft (2004), we adopt the following definition for the set of admissible strategies (see Kraft (2004, p.18)). In this definition, the following two boundedness conditions hold under the worst-case probability measure $\mathbb{Q}^*$. This measure will be determined by solving the following optimization problem (17), since it contains an infimum over all measure $\mathbb{Q} \in \mathcal{Q}$. Due to that each $\mathbb{Q}$ is defined by $\zeta(t)$ in Eq. (13) with $\gamma(t)$, We can select suitable $\gamma^*(t)$ to achieve the minimum, and denote the corresponding measure as $\mathbb{Q}^*$ (Gu et al., 2018).

**Definition 2.1** (*Admissible Strategy*). A trading strategy $\{u(t)\}_{t\in[0,T]}$ is said to be admissible, if

(i) $u(t)$ is a progressively measurable process taking value in $U = \mathbb{R} \times [0,1]^z$ and $\mathbb{E}^{\mathbb{Q}^*}_{t,x,\alpha}[\int_0^T \|u(t)\|^4 dt] < \infty$, where $\mathbb{Q}^*$ is the chosen probability measure to describe the worst case and will be shown later;

(ii) For $\forall(t,x,\alpha) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+$, Eq. (16) has a pathwise unique solution $\{X^u(t)\}_{t\in[0,T]}$ with $\mathbb{E}^{\mathbb{Q}^*}_{t,x,\alpha}|U(X^u(T))| < +\infty$, where $\mathbb{E}^{\mathbb{Q}^*}_{t,x,\alpha}[\cdot] = \mathbb{E}^{\mathbb{Q}^*}[\cdot|X^u(t) = x, \alpha(t) = \alpha]$.

Denote by $\mathcal{U}$ the set of all admissible strategies. Similarly, we can obtain the set of admissible strategies $\tilde{\mathcal{U}}$ with no ambiguity aversion.

To solve the following problem (17), we firstly determine the worst-case measure $\mathbb{Q}^*$ by solving the inner infimum problem, and then obtain the optimal strategies by solving the outer supremum problem under the measure $\mathbb{Q}^*$. So in Definition 2.1, we give the assumption that the boundedness conditions only need to hold under the measure $\mathbb{Q}^*$.

In this paper, we assume that the insurer is aware of the difficulty to get the precise model parameters which are estimated under the real measure $\mathbb{P}$, and she is skeptical about the parameters. In this case, we call the insurer to be ambiguity-averse, she faces the robust optimal investment and reinsurance problem under the alternative measure $\mathbb{Q}$. This means that she finds a robust optimal control which is the best choice in some worst-case models. If the insurer's investment objective is to maximize the exponential utility of the terminal surplus of the company, following Branger and Larsen (2013) and Maenhout (2004), we assume that the insurer faces the following modified optimization problem

$$\sup_{u\in\mathcal{U}} \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}\Big[U(X^u(T)) + \int_0^T \Psi(s)ds\Big], \tag{17}$$

where $\Psi(s) := \frac{\gamma_0^2(s)}{2\psi_0(s)} + \frac{\gamma_1^2(s)}{2\psi_1(s)} + \frac{\gamma_2^2(s)}{2\psi_2(s)}$. The expectation is calculated under the alternative measure $\mathbb{Q}$ defined by $\gamma(t)$, the insurer determines the measure $\mathbb{Q}$ by minimizing the expected utility which means that the worst case is considered. We mark the chosen measure which describes the worst-case as $\mathbb{Q}^*$. At the same time, there is a penalty for moving away from the reference model, which is given by the second term of the expectation. This distance is measured by the weighted relative entropy arising from diffusion risk. By the similar calculation of Branger and Larsen (2013), the increase in relative entropy from $t$ to $t + dt$ equals

$$\frac{1}{2}[\gamma_0^2(t) + \gamma_1^2(t) + \gamma_2^2(t)]dt. \tag{18}$$

The functions $\psi_0(s)$, $\psi_1(s)$ and $\psi_2(s)$ capture the insurer's ambiguity aversion degree with respect to (w.r.t.) the diffusion risk from the insurance market, risky asset and the stochastic factor process. These functions measure the strength of the preference for robustness and are assumed to be nonnegative.

For convenience of analysis, refer to the method of Maenhout (2004), we assume that the preference functions $\psi_0(t)$, $\psi_1(t)$ and $\psi_2(t)$ are state dependent and have the form

$$\psi_0(t) = -\frac{\beta_0}{mV(t,x,\alpha)}, \quad \psi_1(t) = -\frac{\beta_1}{mV(t,x,\alpha)},$$
$$\psi_2(t) = -\frac{\beta_2}{mV(t,x,\alpha)}, \tag{19}$$

where $\beta_0$, $\beta_1$ and $\beta_2$ are nonnegative parameters representing the ambiguity-averse level of insurer to the diffusion risk from the insurance market, risky asset and the stochastic factor process, respectively. Now we define the insurer's indirect utility function by

$$V(t,x,\alpha) = \sup_{u\in\mathcal{U}} \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}_{t,x,\alpha}\Big[U(X^u(T)) + \int_t^T \Psi(s)ds\Big]. \tag{20}$$

## 3. Explicit solution for the problem

### 3.1. Robust optimal reinsurance and investment strategies

In this section, we try to solve the optimization problem (17). Notice that the control $u(t) = (\pi(t), q(t))$ and the processes $\gamma(t) = (\gamma_0(t), \gamma_1(t), \gamma_2(t))$ take values in the value spaces given by $U = \mathbb{R} \times [0,1]^z$ and $\mathbb{R}^3$ respectively. According to the principle of dynamic programming and following the method developed by Anderson et al. (2003), we obtain the robust Hamilton–Jacobi–Bellman (HJB) equation for the optimization problem (17) as follows (for the sake of brevity, we omit the variable $(t,x,\alpha)$ in some functions such as $V$, $\psi_i$ for $i = 0, 1, 2$, and $\mu$):

$$\begin{aligned}
&\sup_{u\in U} \inf_{(\gamma_0,\gamma_1,\gamma_2)\in\mathbb{R}^3} \Big\{ V_t + V_x[c - \delta(q)\\
&- \textstyle\sum_{l=1}^z a_l q_l - \gamma_0\sqrt{\sum_{l=1}^z \sigma_l^2 q_l^2 + \sum_{i\neq j}^z q_i q_j \lambda \mu_i \mu_j}\\
&+\pi(\mu - r - \gamma_1 \frac{\mu-r}{\theta\sqrt{\alpha}}) + rx] + V_\alpha[k(\phi - \alpha)\\
&-\sqrt{\alpha}(k_1\gamma_1 + k_2\gamma_2)] + \tfrac{1}{2}V_{xx}\Big[\textstyle\sum_{l=1}^z \sigma_l^2 q_l^2\\
&+ \textstyle\sum_{i\neq j}^z q_i q_j \lambda \mu_i \mu_j + \frac{(\mu-r)^2}{\theta^2\alpha}\pi^2\Big]\\
&+\tfrac{1}{2}\alpha(k_1^2 + k_2^2)V_{\alpha\alpha} + \tfrac{k_1\pi}{\theta}(\mu - r)V_{x\alpha}\\
&+\tfrac{1}{2}\Big[\tfrac{\gamma_0^2}{\psi_0} + \tfrac{\gamma_1^2}{\psi_1} + \tfrac{\gamma_2^2}{\psi_2}\Big]\Big\} = 0,
\end{aligned} \tag{21}$$

for $\forall\ (t,x,\alpha) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+$ and the terminal condition $V(T,x,\alpha) = -\frac{1}{m}e^{-mx}$. Now we aim to derive the solution to the HJB Eq. (21). Inspired by Zheng et al. (2016), we conjecture that the solution of Eq. (21) is specified by the following form

$$H(t,x,\alpha) = -\frac{1}{m}\exp\{-mxe^{r(T-t)} + M(t) + N(t)\alpha\}, \tag{22}$$

where $M(t)$ and $N(t)$ are only functions of time $t$, which will need to be determined later with the boundary conditions $M(T) = 0$, $N(T) = 0$. Then the corresponding partial derivatives are given by

$$\begin{aligned}
&H_t = [mxre^{r(T-t)} + M'(t) + N'(t)\alpha]H,\\
&\quad H_x = -me^{r(T-t)}H, \quad H_{xx} = m^2e^{2r(T-t)}H;\\
&H_\alpha = N(t)H, \quad H_{x\alpha} = -me^{r(T-t)}N(t)H, \quad H_{\alpha\alpha} = N^2(t)H.
\end{aligned}$$

We first fix $u$ and take the derivative of the inside of $\{\cdot\}$ in Eq. (21) w.r.t. $\gamma_0, \gamma_1, \gamma_2$, respectively. Based on the first-order condition, we obtain the minimum point $\gamma^*(t,\alpha) = (\gamma_0^*(t), \gamma_1^*(t,\alpha), \gamma_2^*(t,\alpha))$ as:

$$\begin{cases}
\gamma_0^*(t) = \beta_0 e^{r(T-t)}\sqrt{\sum_{l=1}^z \sigma_l^2 q_l^2 + \sum_{i\neq j}^z q_i q_j \lambda \mu_i \mu_j},\\
\gamma_1^*(t,\alpha) = \beta_1 e^{r(T-t)}\pi(t)\frac{\mu(t)-r}{\theta\sqrt{\alpha}} - \frac{\beta_1}{m}k_1 N(t)\sqrt{\alpha},\\
\gamma_2^*(t,\alpha) = -\frac{\beta_2}{m}k_2 N(t)\sqrt{\alpha},
\end{cases} \tag{23}$$

which means that $\gamma^*(t,\alpha)$ minimizes the term in $\{\cdot\}$ when $u$ is fixed. Substituting Eq. (23) and the partial derivatives of $H(t,x,\alpha)$

into Eq. (21), we obtain

$$\begin{aligned}&M'(t)+\alpha N'(t)-mce^{r(T-t)}+k(\phi-\alpha)N(t)\\&+\frac{1}{2}\alpha[(\frac{\beta_1}{m}+1)k_1^2+(\frac{\beta_2}{m}+1)k_2^2]N^2(t)\\&+\inf_{\pi\in\mathbb{R}}\Big\{\frac{1}{2}m(m+\beta_1)e^{2r(T-t)}\frac{(\mu-r)^2}{\theta^2\alpha}\pi^2\\&\quad-\frac{(m+\beta_1)k_1}{\theta}(\mu-r)e^{r(T-t)}N(t)\pi\\&-me^{r(T-t)}(\mu-r)\pi\Big\}+\inf_{q\in[0,1]^z}\{G(q)\}=0,\end{aligned}\tag{24}$$

where $G(q)=me^{r(T-t)}[\delta(q)+\sum_{l=1}^z a_lq_l]+\frac{1}{2}m(m+\beta_0)e^{2r(T-t)}(\sum_{l=1}^z\sigma_l^2q_l^2+\sum_{i\neq j}^z q_iq_j\lambda\mu_i\mu_j)$.

Now we take the derivative of the inside of the first $\{\cdot\}$ in Eq. (24) w.r.t. $\pi$, which ensures that the minimize point is

$$\begin{aligned}\pi^*(t,\alpha)&=\frac{\theta^2\alpha}{\mu(t)-r}e^{-r(T-t)}[\frac{1}{m+\beta_1}+\frac{k_1}{m\theta}N(t)]\\&=\frac{\theta\sqrt{\alpha}}{\sigma(t)}e^{-r(T-t)}[\frac{1}{m+\beta_1}+\frac{k_1}{m\theta}N(t)].\end{aligned}\tag{25}$$

Next, we take the derivative of the inside of the second $\{\cdot\}$ in Eq. (24) w.r.t $q$. For any $t\in[0,T]$, we get

$$\begin{cases}\frac{\partial G}{\partial q_l}=me^{r(T-t)}[-\eta a_l-2(1+\eta)\xi((1-q_l)\sigma_l^2\\\quad+\sum_{j=1,j\neq l}^z(1-q_j)\lambda\mu_l\mu_j)]\\\quad\quad+m(m+\beta_0)e^{2r(T-t)}(\sigma_l^2q_l+\sum_{j=1,j\neq l}^z q_j\lambda\mu_l\mu_j),\\\frac{\partial^2G}{\partial q_l^2}=me^{r(T-t)}\sigma_l^2[2\xi(1+\eta)+(m+\beta_0)e^{r(T-t)}],\\\frac{\partial^2G}{\partial q_l\partial q_k}=\frac{\partial^2G}{\partial q_k\partial q_l}=me^{r(T-t)}\lambda\mu_l\mu_k[2\xi(1+\eta)+(m+\beta_0)e^{r(T-t)}],\end{cases}$$

for $l\neq k$, $l,k=1,2,\dots,z$. So $\begin{bmatrix}\frac{\partial^2G}{\partial q_1^2}&\frac{\partial^2G}{\partial q_1\partial q_2}&\cdots&\frac{\partial^2G}{\partial q_1\partial q_z}\\\frac{\partial^2G}{\partial q_2\partial q_1}&\frac{\partial^2G}{\partial q_2^2}&\cdots&\frac{\partial^2G}{\partial q_2\partial q_z}\\\vdots&\vdots&\ddots&\vdots\\\frac{\partial^2G}{\partial q_z\partial q_1}&\frac{\partial^2G}{\partial q_z\partial q_2}&\cdots&\frac{\partial^2G}{\partial q_z^2}\end{bmatrix}=$ $me^{r(T-t)}[2\xi(1+\eta)+(m+\beta_0)e^{r(T-t)}]\cdot\mathbf{A}$, where

$$\mathbf{A}=\begin{bmatrix}\sigma_1^2&\lambda\mu_1\mu_2&\lambda\mu_1\mu_3&\dots&\lambda\mu_1\mu_z\\\lambda\mu_2\mu_1&\sigma_2^2&\lambda\mu_2\mu_3&\dots&\lambda\mu_2\mu_z\\\vdots&\vdots&\vdots&\ddots&\vdots\\\lambda\mu_z\mu_1&\lambda\mu_z\mu_2&\lambda\mu_z\mu_3&\dots&\sigma_z^2\end{bmatrix}.$$

By the Lemma 1 of Yuen et al. (2015), $\mathbf{A}$ is a positive definite matrix, which means that the Hessian matrix is also a positive definite matrix. Thus, $G(q)$ is a convex function w.r.t. $q_1,\dots,q_z$. Therefore, using the first-order optimization conditions, the minimizer $\hat{q}=(\hat{q}_1,\hat{q}_2,\dots,\hat{q}_z)^\top$ for $G(q)$ satisfies the following equation

$$[2\xi(1+\eta)+(m+\beta_0)e^{r(T-t)}]\mathbf{A}\cdot\hat{q}=2\xi(1+\eta)\mathbf{A}\cdot\mathbf{1}+\eta\cdot\mathbf{a},\tag{26}$$

where $\mathbf{1}^\top=(1,1,\dots,1)_{1\times z}$, $\mathbf{a}=(a_1,a_2,\dots,a_z)^\top$. Due to the fact that $\mathbf{A}$ is positive definite matrix, which guarantees the invertibility of this matrix. Thus the minimizer $\hat{q}(t)$ is

$$\hat{q}(t)=\frac{2\xi(1+\eta)\mathbf{1}+\eta\mathbf{A}^{-1}\cdot\mathbf{a}}{2\xi(1+\eta)+(m+\beta_0)e^{r(T-t)}}.\tag{27}$$

In Eq. (27), when $\eta=0$, the reinsurance premium degenerates to the variance premium rate. In this case, the reinsurance proportion for the $l$th line of business is $\hat{q}_l(t)=\frac{2\xi}{2\xi+(m+\beta_0)e^{r(T-t)}}$, $l=1,2,\dots,z$, which satisfies the condition $\hat{q}_l(t)\in[0,1]$ and as a result it is the optimal reinsurance strategy. When $\eta\neq0$, to make sure that the optimal reinsurance proportions satisfy $q_l(t)\in[0,1]$, for $l=1,2,\dots,z$, we need to investigate the optimal strategy in different cases for the value of the parameters. Here, for simplicity, we only consider the optimal reinsurance proportion under the case of $z=2$, i.e., the insurance company has two lines of business. The following ideas and methods are still useful for deriving optimal results when $z>2$.[1]

Let

$$\begin{aligned}&U_1=a_1\sigma_2^2-a_2\lambda\mu_1\mu_2,\quad U_2=a_2\sigma_1^2-a_1\lambda\mu_1\mu_2,\\&U_3=\sigma_1^2\sigma_2^2-\lambda^2\mu_1^2\mu_2^2.\end{aligned}\tag{28}$$

For $z=2$, Eq. (27) becomes

$$\begin{cases}\hat{q}_1(t)=\frac{\eta U_1+2(1+\eta)\xi U_3}{[2\xi(1+\eta)+(m+\beta_0)e^{r(T-t)}]U_3},\\\hat{q}_2(t)=\frac{\eta U_2+2(1+\eta)\xi U_3}{[2\xi(1+\eta)+(m+\beta_0)e^{r(T-t)}]U_3}.\end{cases}\tag{29}$$

Note that $U_1,U_2,U_3>0$, thus $\hat{q}_1(t)>0$, $\hat{q}_2(t)>0$. Furthermore, let

$$t_1=\begin{cases}T,&\eta U_1\le(m+\beta_0)U_3,\\T-\frac{1}{r}\ln\frac{\eta U_1}{(m+\beta_0)U_3},&(m+\beta_0)U_3<\eta U_1<(m+\beta_0)U_3e^{rT},\\0,&\eta U_1\ge(m+\beta_0)U_3e^{rT},\end{cases}\tag{30}$$

and

$$t_2=\begin{cases}T,&\eta U_2\le(m+\beta_0)U_3,\\T-\frac{1}{r}\ln\frac{\eta U_2}{(m+\beta_0)U_3},&(m+\beta_0)U_3<\eta U_2<(m+\beta_0)U_3e^{rT},\\0,&\eta U_2\ge(m+\beta_0)U_3e^{rT}.\end{cases}\tag{31}$$

We now discuss the optimal reinsurance strategy under following different cases.

(1) If $U_1\le U_2$, then $t_1\ge t_2$, so

(i) When $0\le t<t_2$, the optimal reinsurance proportion $q^*(t)=(q_1^*(t),q_2^*(t))=(\hat{q}_1(t),\hat{q}_2(t))$, where, $\hat{q}_1(t)$ and $\hat{q}_2(t)$ are given by (29).

(ii) When $t\ge t_2$, we have $\hat{q}_2(t)\ge1$, so $q_2^*(t)=1$. Substituting $q_2^*(t)=1$ into the second $\{\cdot\}$ of Eq. (24) yields the following optimization problem:

$$\begin{aligned}&\inf_{q_1}\Big\{me^{r(T-t)}[(1+\eta)((1-q_1)a_1+\xi(1-q_1)^2\sigma_1^2)+a_1q_1+a_2]\\&\quad+\frac{1}{2}m(m+\beta_0)e^{2r(T-t)}[\sigma_1^2q_1^2+\sigma_2^2+2\lambda q_1\mu_1\mu_2]\Big\}.\end{aligned}\tag{32}$$

For $t\le T$, it can be shown that the minimizer of problem (32) has the form

$$\bar{q}_1(t)=\frac{\eta a_1+2\xi(1+\eta)\sigma_1^2-\lambda\mu_1\mu_2(m+\beta_0)e^{r(T-t)}}{\sigma_1^2[2\xi(1+\eta)+(m+\beta_0)e^{r(T-t)}]}.\tag{33}$$

Let

$$t_1'=T-\frac{1}{r}\ln\frac{\eta a_1}{(m+\beta_0)(\sigma_1^2+\lambda\mu_1\mu_2)}.\tag{34}$$

Then for $t_2\le t\le t_1'$, the optimal reinsurance strategy is $(q_1^*(t),q_2^*(t))=(\bar{q}_1(t),1)$.

(iii) For $t_1'\le t\le T$, it is easy to see that $(q_1^*(t),q_2^*(t))=(1,1)$.

(2) If $U_1>U_2$, then $t_1<t_2$, so

[1] Since when $z$ gets bigger, the possible cases will increase geometrically, which leads to many cases need to be discussed, and the analysis process becomes challenging and complicated. We can only discuss the solution case by case when $z>2$.

(i) When $0 \le t < t_1$, we have $(q_1^*(t), q_2^*(t)) = (\hat{q}_1(t), \hat{q}_2(t))$, where, $\hat{q}_1(t)$ and $\hat{q}_2(t)$ are given by (29).

(ii) When $t \ge t_1$, we have $\hat{q}_1(t) > 1$, so $q_1^*(t) = 1$. Substituting $q_1^*(t) = 1$ into the second $\{\cdot\}$ of Eq. (24), we obtain the following optimization problem:

$$\inf_{q_2}\Big\{me^{r(T-t)}[(1+\eta)((1-q_2)a_2+\xi(1-q_2)^2\sigma_2^2)+a_1+a_2q_2] + \frac{1}{2}m(m+\beta_0)e^{2r(T-t)}[\sigma_1^2+\sigma_2^2q_2^2+2\lambda q_2\mu_1\mu_2]\Big\}. \tag{35}$$

For $t \le T$, the minimizer of problem (35) has the form

$$\bar{q}_2(t) = \frac{\eta a_2 + 2\xi(1+\eta)\sigma_2^2 - \lambda\mu_1\mu_2(m+\beta_0)e^{r(T-t)}}{\sigma_2^2[2\xi(1+\eta)+(m+\beta_0)e^{r(T-t)}]}. \tag{36}$$

Let

$$t_2' = T - \frac{1}{r}\ln\frac{\eta a_2}{(m+\beta_0)(\sigma_2^2+\lambda\mu_1\mu_2)}. \tag{37}$$

Then for $t_1 \le t < t_2'$, the optimal reinsurance strategy is $(q_1^*(t), q_2^*(t)) = (1, \bar{q}_2(t))$.

(iii) For $t_2' \le t \le T$, it is easy to see that $(q_1^*(t), q_2^*(t)) = (1, 1)$.

Now substituting $q^*(t)$ and $\pi^*(t)$ into Eq. (24), and separating the variables with and without $\alpha$, respectively, we derive the following equations

$$N'(t) - (k+\theta k_1)N(t) + \frac{m+\beta_2}{2m}k_2^2N^2(t) - \frac{m\theta^2}{2(m+\beta_1)} = 0, \tag{38}$$

$$M'(t) - mce^{r(T-t)} + k\phi N(t) + G(q_1^*(t), q_2^*(t)) = 0 \tag{39}$$

with the boundary conditions $N(T) = 0$ and $M(T) = 0$. Solving Eq. (38), we obtain

$$N(t) = \frac{b_3[1-e^{b_1(T-t)}]}{2b_1+(b_1+b_2)[e^{b_1(T-t)}-1]}, \tag{40}$$

where

$$b_1 = \sqrt{(k+\theta k_1)^2 + \frac{m+\beta_2}{m+\beta_1}k_2^2\theta^2}, \quad b_2 = k+\theta k_1, \quad b_3 = \frac{m\theta^2}{m+\beta_1}. \tag{41}$$

To solve Eq. (39), we should discuss the solution under the following cases.

**Case 1:** $U_1 \le U_2$.

(i) When $0 \le t < t_2$, we get

$$M(t) = M_1(t) = \int_t^T [-mce^{r(T-s)} + k\phi N(s) + G(\hat{q}_1(s), \hat{q}_2(s))]ds + c_1, \tag{42}$$

where $c_1$ is a constant that will be determined later.

(ii) When $t_2 \le t < t_1'$, we have

$$M(t) = M_2(t) = \int_t^T [-mce^{r(T-s)} + k\phi N(s) + G(\bar{q}_1(s), 1)]ds + c_2, \tag{43}$$

where the constant $c_2$ also will be determined later.

(iii) When $t_1' \le t \le T$, we get

$$M(t) = M_3(t) = \int_t^T [-mce^{r(T-s)} + k\phi N(s) + G(1, 1)]ds. \tag{44}$$

To ensure that $H(t, x, \alpha)$ is the classical solution of HJB Eq. (21), we require $H(t, x, \alpha) \in C^{1,2,2}$ for any $(t, x, \alpha) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+$, which means $M_1(t_2) = M_2(t_2)$, $M_2(t_1') = M_3(t_1')$, $M_1'(t_2) = M_2'(t_2)$ and $M_2'(t_1') = M_3'(t_1')$ must hold. Thus the constants $c_1$ and $c_2$ are given by $c_1 = \int_{t_2}^T [G(\bar{q}_1(s), 1) - G(\hat{q}_1(s), \hat{q}_2(s))]ds + c_2$ and $c_2 = \int_{t_1'}^T [G(1, 1) - G(\bar{q}_1(s), 1)]ds$.

**Case 2:** $U_1 > U_2$.

(i) When $0 \le t < t_1$, we get

$$M(t) = M_4(t) = \int_t^T [-mce^{r(T-s)} + k\phi N(s) + G(\hat{q}_1(s), \hat{q}_2(s))]ds + c_3, \tag{45}$$

where $c_3$ is a constant that will be determined later.

(ii) When $t_1 \le t \le t_2'$, we have

$$M(t) = M_5(t) = \int_t^T [-mce^{r(T-s)} + k\phi N(s) + G(1, \bar{q}_2(s))]ds + c_4, \tag{46}$$

where $c_4$ is a constant that will be determined later.

(iii) When $t_2' < t \le T$, we get

$$M(t) = M_3(t) = \int_t^T [-mce^{r(T-s)} + k\phi N(s) + G(1, 1)]ds. \tag{47}$$

To ensure that $H(t, x, \alpha) \in C^{1,2,2}$ for any $(t, x, \alpha) \in [0, T]\times\mathbb{R}\times\mathbb{R}^+$, the conditions $M_4(t_1) = M_5(t_1)$, $M_3(t_2') = M_5(t_2')$, $M_4'(t_1) = M_5'(t_1)$ and $M_3'(t_2') = M_5'(t_2')$ should be satisfied. Thus, we derive $c_3 = \int_{t_1}^T [G(1, \bar{q}_2(s)) - G(\hat{q}_1(s), \hat{q}_2(s))]ds + c_4$ and $c_4 = \int_{t_2'}^T [G(1, 1) - G(1, \bar{q}_2(s))]ds$.

Summarizing the above results, we obtain the following verification theorem.

**Theorem 3.1.** *For the optimal control problem (17) with $z = 2$, if the parameters satisfy technical condition (54) and*

$$\begin{cases} \frac{24m^2\theta^2}{(m+\beta_1)^2} - \frac{56m\theta k_1 b_3}{(m+\beta_1)(b_1+b_2)} + \frac{32k_1^2b_3^2}{(b_1+b_2)^2} \le \frac{k^2}{2(k_1^2+k_2^2)}, \\ \frac{24m^2\theta^2}{(m+\beta_1)^2} \le \frac{k^2}{2(k_1^2+k_2^2)}. \end{cases} \tag{48}$$

*with $b_1$, $b_2$ and $b_3$ given in (41). And let $\hat{q}_1(t)$, $\hat{q}_2(t)$, $\bar{q}_1(t)$, $\bar{q}_2(t)$ be given in (29), (33) and (36), respectively. Then,*

*(1) When $U_1 \le U_2$ (see Eq. (28)), the robust optimal reinsurance strategies of an AAI with exponential utility function are*

$$(q_1^*(t), q_2^*(t)) = \begin{cases} (\hat{q}_1(t), \hat{q}_2(t)), & 0 \le t < t_2, \\ (\bar{q}_1(t), 1), & t_2 \le t < t_1', \\ (1, 1), & t_1' \le t < T, \end{cases} \tag{49}$$

*with $t_2$ and $t_1'$ given in Eqs. (31) and (34) for any $t \in [0, T]$. The value function is given by*

$$V(t, x, \alpha) = H(t, x, \alpha) = \begin{cases} -\frac{1}{m}\exp\{-mxe^{r(T-t)} + M_1(t) + N(t)\alpha\}, & 0 \le t < t_2, \\ -\frac{1}{m}\exp\{-mxe^{r(T-t)} + M_2(t) + N(t)\alpha\}, & t_2 \le t < t_1', \\ -\frac{1}{m}\exp\{-mxe^{r(T-t)} + M_3(t) + N(t)\alpha\}, & t_1' \le t < T, \end{cases} \tag{50}$$

*with $N(t)$ and $M_i(t)$, $i = 1, 2, 3$ given in Eqs. (40)–(44), respectively.*

*(2) When $U_1 > U_2$, the robust optimal reinsurance strategies of an AAI with exponential utility function are*

$$(q_1^*(t), q_2^*(t)) = \begin{cases} (\hat{q}_1(t), \hat{q}_2(t)), & 0 \le t < t_1, \\ (1, \bar{q}_2(t)), & t_1 \le t < t_2', \\ (1, 1), & t_2' \le t < T, \end{cases} \tag{51}$$

*with $t_1$ and $t_2'$ given in Eqs.* (30) and (37) *for any $t \in [0, T]$. The value function is given by*

$$V(t,x,\alpha)=H(t,x,\alpha)=\begin{cases}-\frac{1}{m}\exp\{-mxe^{r(T-t)}\\ \quad +M_4(t)+N(t)\alpha\}, \quad 0\le t<t_1,\\ -\frac{1}{m}\exp\{-mxe^{r(T-t)}\\ \quad +M_5(t)+N(t)\alpha\}, \quad t_1\le t<t_2',\\ -\frac{1}{m}\exp\{-mxe^{r(T-t)}\\ \quad +M_3(t)+N(t)\alpha\}, \quad t_2'\le t<T,\end{cases} \tag{52}$$

*with $M_4(t)$ and $M_5(t)$ given in Eqs.* (45) *and* (46)*, respectively.*

*In both cases, the robust optimal investment strategy is given in Eq.* (25) *by replacing $\alpha$ with the stochastic process $\alpha(t)$. The worst-case $\gamma^*(t,\alpha(t))$ is given by*

$$\begin{cases}\gamma_0^*(t)=\beta_0 e^{r(T-t)}\sqrt{\sigma_1^2 q_1^*(t)^2+\sigma_2^2 q_2^*(t)^2+2q_1^*(t)q_2^*(t)\lambda\mu_1\mu_2},\\ \gamma_1^*(t,\alpha(t))=\frac{\beta_1\theta}{m+\beta_1}\sqrt{\alpha(t)},\\ \gamma_2^*(t,\alpha(t))=-\frac{\beta_2}{m}N(t)k_2\sqrt{\alpha(t)}.\end{cases} \tag{53}$$

**Proof.** See Appendix. □

**Remark 3.1.** The Novikov's condition holds for $\gamma^*(t,\alpha(t)) = (\gamma_0^*(t), \gamma_1^*(t,\alpha(t)), \gamma_2^*(t,\alpha(t)))$, if the parameters satisfy the condition:

$$\frac{\beta_1^2\theta^2}{(m+\beta_1)^2}+\frac{\beta_2^2k_2^2n^2}{m^2}<\frac{k^2}{k_1^2+k_2^2}, \tag{54}$$

where $n=\frac{b_3}{b_1+b_2}$. This condition can be induced by the similar method in Corollary 4.1 of Yi et al. (2013), here we omit this process.

**Remark 3.2.** Based on Eq. (25), we find that the robust optimal investment strategy is Markovian on $\alpha(t)$ and independent of the parameters in insurance market, which depends only on the interest rate, ambiguity-aversion coefficient, and parameters based on the financial market. From Eqs. (49) and (51), we find that the optimal proportional reinsurance strategy is a deterministic function on time $t$ and also independent of the price parameters of the risky asset. Besides, the robustness on the optimal strategy just enlarges the insurer's risk aversion level under the exponential utility. Thus, the ambiguity aversion level can be understood as an extra risk aversion parameter. Similar results also are shown in Maenhout (2004) and Yi et al. (2013) under the power and exponential utility functions, respectively. It is worth pointing out that, under the framework of game theory, Pun (2018) investigates the robust time-inconsistent portfolio problem with state-dependent risk aversion. He provides an example that under the mean–variance criteria the robustness effect on the equilibrium strategy is complicated, in which the equilibrium strategy has highly nonlinear relationship with the ambiguity aversion level.

**Remark 3.3.** If the insurance company has only one business, the claim size random variable is $X_i^{(1)}$, and claim intensity is $\lambda_1$ (i.e. $\lambda_2=\cdots=\lambda_m=\lambda=0$), the reinsurance premium is calculated under expectation principle ($\eta>0, \xi=0$) and the reinsurance strategy $q_1(t)\in(0,+\infty)$. Moreover,

(i) if the price process of the risky asset satisfies CEV model (the corresponding parameters are set as those in Remark 2.4), then

$$q_1^*(t)=\frac{a_1\eta}{(m+\beta_0)\sigma_1^2}e^{-r(T-t)}, \tag{55}$$

$$\pi^*(t)=\frac{(\mu-r)+\frac{(\mu-r)^2}{2r}(1-e^{-2\beta r(T-t)})}{(m+\beta_1)\varrho^2S^{2\beta}(t)e^{r(T-t)}}. \tag{56}$$

This result is consistent with the optimal strategy in Zheng et al. (2016), so our model is extension of Zheng et al. (2016).

(ii) if the price process of the risky asset satisfies the Heston's stochastic volatility model (the corresponding parameters are set as those in Remark 2.5), and the ambiguity aversion coefficients $\beta_0=\beta_1=\beta_2$, then the optimal reinsurance strategy is also presented by Eq. (55) and the optimal investment strategy is simplified by

$$\pi^*(t)=e^{-r(T-t)}[\frac{\theta}{m+\beta_1}+\frac{\sigma_0\rho}{m}N(t)]. \tag{57}$$

In this case, the optimal strategy in our model is degenerated to the optimal strategy in Yi et al. (2013).

### 3.2. The case of ambiguity-neutral

In this subsection, we assume that the insurer is ambiguity-neutral, which means that all the ambiguity-aversion coefficients satisfy $\beta_0=\beta_1=\beta_2=0$. Under the real measure $\mathbb{P}$, the surplus process can be described by Eq. (11). Meanwhile, the indirect utility function is defined by $\widetilde{V}(t,x,\alpha)=\sup_{u\in\tilde{\mathcal{U}}}\mathbb{E}^{\mathbb{P}}_{t,x,\alpha}[-\frac{1}{m}e^{-mX^u(T)}]$. Then the corresponding HJB equation is

$$\sup_{u\in U}\Big\{\widetilde{V}_t+\widetilde{V}_x[c-\delta(q)-\textstyle\sum_{l=1}^z a_lq_l \\ +\pi(\mu-r)+rx]+\widetilde{V}_\alpha[k(\phi-\alpha)] \\ +\frac{1}{2}\widetilde{V}_{xx}[\sum_{l=1}^z\sigma_l^2q_l^2+\sum_{i\neq j}^z q_iq_j\lambda\mu_i\mu_j+\frac{(\mu-r)^2}{\theta^2\alpha}\pi^2] \\ +\frac{1}{2}\alpha(k_1^2+k_2^2)\widetilde{V}_{\alpha\alpha}+\frac{k_1\pi}{\theta}(\mu-r)\widetilde{V}_{x\alpha}\Big\}=0. \tag{58}$$

For $z=2$, using the method similar to solve HJB equation (21) and Verification Theorem similar to Theorem 3.1, the optimal investment strategy is

$$\tilde{\pi}^*(t,\alpha(t))=\frac{\theta^2\alpha(t)}{\mu(t)-r}e^{-r(T-t)}[\frac{1}{m}+\frac{k_1}{m\theta}\tilde{N}(t)], \tag{59}$$

where

$$\tilde{N}(t)=\frac{\tilde{b}_3[e^{1-\tilde{b}_1(T-t)}]}{2\tilde{b}_1+(\tilde{b}_1+\tilde{b}_2)[e^{\tilde{b}_1(T-t)}-1]}, \\ \tilde{b}_1=\sqrt{(k+\theta k_1)^2+k_2^2\theta^2}, \tilde{b}_2=b_2, \tilde{b}_3=\theta^2. \tag{60}$$

Let

$$\tilde{t}_1=\begin{cases}T, & \eta U_1\le mU_3,\\ T-\frac{1}{r}\ln\frac{\eta U_1}{mU_3}, & mU_3<\eta U_1<mU_3e^{rT},\\ 0, & \eta U_1\ge mU_3e^{rT},\end{cases}$$

and

$$\tilde{t}_2=\begin{cases}T, & \eta U_2\le mU_3,\\ T-\frac{1}{r}\ln\frac{\eta U_2}{mU_3}, & mU_3<\eta U_2<mU_3e^{rT},\\ 0, & \eta U_2\ge mU_3e^{rT}.\end{cases}$$

Let $\tilde{t}_1'=T-\frac{1}{r}\ln\frac{\eta a_1}{m(\sigma_1^2+\lambda\mu_1\mu_2)}$, $\tilde{t}_2'=T-\frac{1}{r}\ln\frac{\eta a_2}{m(\sigma_2^2+\lambda\mu_1\mu_2)}$, $\tilde{q}_1(t)=\hat{q}_1(t)|_{\beta_0=0}$, $\tilde{q}_2(t)=\hat{q}_2(t)|_{\beta_0=0}$, $\bar{\bar{q}}_1(t)=\bar{q}_1(t)|_{\beta_0=0}$, $\bar{\bar{q}}_2(t)=\bar{q}_2(t)|_{\beta_0=0}$ and $\tilde{G}(q_1(t),q_2(t))=G(q_1(t),q_2(t))|_{\beta_0=0}$. Then the optimal reinsurance strategy $\tilde{q}^*(t)=(\tilde{q}_1^*(t),\tilde{q}_2^*(t))$, and corresponding value function $\widetilde{V}(t,x,\alpha)$ are given as follows:

(1) If $U_1 \le U_2$, the optimal reinsurance strategies are

$$(\tilde{q}_1^*(t), \tilde{q}_2^*(t)) = \begin{cases} (\tilde{q}_1(t), \tilde{q}_2(t)), & 0 \le t < \tilde{t}_2, \\ (\bar{\tilde{q}}_t(t), 1), & \tilde{t}_2 \le t < \tilde{t}_1', \\ (1, 1), & \tilde{t}_1' \le t < T, \end{cases} \tag{61}$$

for any $t \in [0, T]$ and the value function is given by

$$\widetilde{V}(t,x,\alpha) = \begin{cases} -\frac{1}{m}\exp\{-mxe^{r(T-t)} + \widetilde{M}_1(t) + \tilde{N}(t)\alpha\}, \\ 0 \le t < \tilde{t}_2, \\ -\frac{1}{m}\exp\{-mxe^{r(T-t)} + \widetilde{M}_2(t) + \tilde{N}(t)\alpha\}, \\ \tilde{t}_2 \le t < \tilde{t}_1', \\ -\frac{1}{m}\exp\{-mxe^{r(T-t)} + \widetilde{M}_3(t) + \tilde{N}(t)\alpha\}, \\ \tilde{t}_1' \le t < T, \end{cases} \tag{62}$$

where

$$\widetilde{M}_1(t) = \int_t^T [-mce^{r(T-s)} + k\phi\tilde{N}(s) + \tilde{G}(\tilde{q}_1(s), \tilde{q}_2(s))]ds + c_5,$$

$$\widetilde{M}_2(t) = \int_t^T [-mce^{r(T-s)} + k\phi\tilde{N}(s) + \tilde{G}(\bar{\tilde{q}}_1(s), 1)]ds + c_6,$$

$$\widetilde{M}_3(t) = \int_t^T [-mce^{r(T-s)} + k\phi\tilde{N}(s) + \tilde{G}(1, 1)]ds,$$

$c_6 = \int_{\tilde{t}_1'}^T [\tilde{G}(1,1) - \tilde{G}(\bar{\tilde{q}}_1(s), 1)]ds$ and $c_5 = \int_{\tilde{t}_2}^T [\tilde{G}(\bar{\tilde{q}}_1(s), 1) - \tilde{G}(\tilde{q}_1(s), \tilde{q}_2(s))]ds + c_6$ to ensure that $\widetilde{V}(t,x,\alpha) \in C^{1,2,2}$ for any $(t,x,\alpha) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+$.

(2) If $U_1 > U_2$, the optimal reinsurance strategies are

$$(\tilde{q}_1^*(t), \tilde{q}_2^*(t)) = \begin{cases} (\tilde{q}_1(t), \tilde{q}_2(t)), & 0 \le t < \tilde{t}_1, \\ (1, \bar{\tilde{q}}_2(t)), & \tilde{t}_1 \le t < \tilde{t}_2', \\ (1, 1), & \tilde{t}_2' \le t < T, \end{cases} \tag{63}$$

for any $t \in [0, T]$ and the value function is given by

$$\widetilde{V}(t,x,\alpha) = \begin{cases} -\frac{1}{m}\exp\{-mxe^{r(T-t)} \\ +\widetilde{M}_4(t) + \tilde{N}(t)\alpha\}, \quad 0 \le t < \tilde{t}_1, \\ -\frac{1}{m}\exp\{-mxe^{r(T-t)} \\ +\widetilde{M}_5(t) + \tilde{N}(t)\alpha\}, \quad \tilde{t}_1 \le t < \tilde{t}_2', \\ -\frac{1}{m}\exp\{-mxe^{r(T-t)} \\ +\widetilde{M}_3(t) + \tilde{N}(t)\alpha\}, \quad \tilde{t}_2' \le t < T, \end{cases} \tag{64}$$

where

$$\widetilde{M}_4(t) = \int_t^T [-mce^{r(T-s)} + k\phi\tilde{N}(s) + \tilde{G}(\tilde{q}_1(s), \tilde{q}_2(s))]ds + c_7,$$

$$\widetilde{M}_5(t) = \int_t^T [-mce^{r(T-s)} + k\phi\tilde{N}(s) + \tilde{G}(1, \bar{\tilde{q}}_2(s))]ds + c_8,$$

$c_8 = \int_{\tilde{t}_2'}^T [\tilde{G}(1,1) - \tilde{G}(1, \bar{\tilde{q}}_2(s))]ds$ and $c_7 = \int_{\tilde{t}_1}^T [\tilde{G}(1, \bar{\tilde{q}}_2(s)) - \tilde{G}(\tilde{q}_1(s), \tilde{q}_2(s))]ds + c_8$ to ensure that $\widetilde{V}(t,x,\alpha) \in C^{1,2,2}$ for any $(t,x,\alpha) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+$.

Comparing Eqs. (59), (61), (63) with Eqs. (25), (49), (51), respectively, we find that the optimal strategy of an ANI is the special case of the strategy for an AAI when the ambiguity-aversion coefficients $\beta_0 = \beta_1 = \beta_2 = 0$. Compared with ANI, the reinsurance retention proportion of the AAI is more lower, which means that the AAI is more conservative.

**Remark 3.4.** When the insurance company has two lines of business, i.e., $z = 2$, the reinsurance premium is calculated under the variance principle ($\eta = 0$), and the insurer is ambiguity-neutral ($\beta_0 = 0$), then Eq. (29) becomes

$$\hat{q}_1(t) = \hat{q}_2(t) = \frac{2\xi}{2\xi + me^{r(T-t)}} \in (0, 1). \tag{65}$$

Thus the optimal reinsurance strategies $q_1^*(t) = q_2^*(t) = \hat{q}_1(t)$, which are consistent with the strategies in Liang and Yuen (2016). In our model, the reinsurance premium is calculated under the generalized mean–variance premium principle. Compared with Eq. (65), we find that the robust optimal reinsurance strategies depend on not only the safety loading, time and interest rate, but also the ambiguity-aversion coefficient, the claim amount and intensity parameters. It is different from the optimal reinsurance strategies under the variance premium principle (Liang and Yuen, 2016).

## 4. Suboptimal strategy

To evaluate the importance of taking ambiguity into account, we determine how much an insurer suffers from ignoring it. Suppose that an AAI does not take the optimal strategy $u^*(t, \alpha(t)) = (\pi^*(t, \alpha(t)), q^*(t))$, but makes her investment decision as if she is an ANI. In other words, she follows the optimal strategy $\tilde{u}^*(t, \alpha(t)) = (\tilde{\pi}^*(t, \alpha(t)), \tilde{q}^*(t))$ given in Section 3.2 (In this section, to conveniently compare with the value function in Section 3.1, we also consider the case of $z = 2$). In this case, we call the strategy as the suboptimal strategy. The aim of this section is to quantify the wealth-equivalent utility loss of an insurer who follows a suboptimal investment and reinsurance strategies. More specifically, we will assess the importance of taking into account model uncertainty in the model. When the AAI adopts the suboptimal strategy, the corresponding value function under the given strategy $\tilde{u}^*$ is defined by

$$\widehat{V}(t,x,\alpha) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{t,x,\alpha}^{\mathbb{Q}}\Big[U(X^{\tilde{u}^*}(T)) + \int_t^T \Psi(s)ds\Big]. \tag{66}$$

We can solve problem (66) using the similar HJB equation as that in Section 3.1. We find that the worst-case measure $\gamma^*(t, \alpha(t))$ for the problem (66) has the similar expression as Eq. (23) only need to change $N(t)$ to function $\hat{N}(t)$. The AAI with ambiguity-aversion coefficients $\beta_i > 0, i = 0, 1, 2$ adopts the strategy $\tilde{u}^*(t, \alpha(t))$ as if she is an ANI under the worst-case measure $\gamma^*(t, \alpha(t))$. Via a calculation that is parallel to the method in Section 3, we obtain the value function for optimization problem (66) under the suboptimal strategy $\tilde{u}^*(t, \alpha(t))$ as follows:

$$\widehat{V}(t,x,\alpha) = -\frac{1}{m}\exp[-mxe^{r(T-t)} + \hat{M}(t) + \hat{N}(t)\alpha], \tag{67}$$

where $\hat{N}(t)$ and $\hat{M}(t)$ satisfy the following ODEs:

$$\begin{aligned} &\hat{N}'(t) - [k + (1 + \frac{\beta_1}{m})k_1\theta(1 + \frac{k_1}{\theta}\tilde{N}(t))]\hat{N}(t) \\ &+ \frac{1}{2}[(1 + \frac{\beta_1}{m})k_1^2 + (1 + \frac{\beta_2}{m})k_2^2]\hat{N}^2(t) \\ &+ \frac{(m + \beta_1)\theta^2}{2m}[1 + \frac{k_1}{\theta}\tilde{N}(t)]^2 - \theta^2[1 + \frac{k_1}{\theta}\tilde{N}(t)] = 0, \end{aligned} \tag{68}$$

$$\hat{M}'(t) - mce^{r(T-t)} + k\phi\hat{N}(t) + G(\tilde{q}_1^*(t), \tilde{q}_2^*(t)) = 0, \tag{69}$$

with boundary conditions $\hat{N}(T) = 0$ and $\hat{M}(T) = 0$, where $\tilde{N}(t)$ is given by (60). Although it is hard to obtain the closed-form solution to the differential equation (68) due to the variable coefficients, we can solve this equation with Runge–Kutta numerical method.

By definition, the suboptimal strategy $\tilde{u}^*(t, \alpha(t))$ will generate a lower expected utility than that by the optimal strategy

**Table 1**
Basic parameters.

| $T$ | $\alpha_1$ | $\alpha_2$ | $\lambda_1$ | $\lambda_2$ | $\lambda$ | $\eta$ | $\xi$ | $r$ | $m$ | $\beta_0$ | $\beta_1$ | $\beta_2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 2 | 3 | 3 | 4 | 1 | 0.3 | 0.5 | 0.05 | 1.2 | 1 | 1 | 1 |

$u^*(t, \alpha(t))$ for the same initial condition. i.e., $\widehat{V}(t, x, \alpha) \leq V(t, x, \alpha)$. In the literature such as Larsen and Munk (2012) and Branger and Larsen (2013), to assess the importance of taking ambiguity into account, they assume that if an investor follows a suboptimal strategy, there will be a wealth-equivalent utility loss. In their models, the power utility is adopted, and the utility loss is measured by the percentage of the initial wealth that the investor is willing to give up to know the optimal strategy. In our model, because of the exponential utility that is adopted, to make the definition of the loss function understandable and simultaneously to make the mathematical calculation easier, we follow the similar idea mentioned in the above literature. When the insurer adopts the suboptimal strategy $\tilde{u}^*(t, \alpha(t))$, we measure the loss as the amount of the initial wealth $x$ that the insurer is willing to give up to know the optimal strategy. Namely, the loss $J$ is defined by

$$V(t, x - J, \alpha) = \widehat{V}(t, x, \alpha). \tag{70}$$

From (22) and (67), it then follows that

$$J \equiv J(t) = \frac{1}{m} e^{-r(T-t)}[(\hat{M}(t) - M(t)) + (\hat{N}(t) - N(t))\alpha(t)]. \tag{71}$$

The loss function $J(t)$ measures the utility loss of the AAI when she adopts the suboptimal strategy $\tilde{u}^*(t, \alpha(t))$.

## 5. Numerical examples

In this section, we present some numerical examples to verify the theoretical results and give some sensitivity analysis on the optimal strategy. In order to analyze conveniently, we only consider that there are two lines of business in the insurance company (i.e., $z = 2$). Similar to Yuen et al. (2015) and Liang and Yuen (2016), we assume that the claim sizes random variables $X_i^{(1)}$ and $X_i^{(2)}$ follow exponential distribution with parameters $\alpha_1$ and $\alpha_2$, respectively. For the price process of the risky asset, we only consider the CEV model and the Heston's model as two special cases. Unless otherwise stated, the basic parameters are given in Table 1.

Besides, in the CEV model, the parameters are given by $s_0 = 0.5, \mu = 0.12, \varrho = 0.2, \beta = 0.3$. In the Heston's model, the related parameters are given by $\sigma_0 = 0.3, \rho = 0.3, \theta = 0.5, k = 2, \phi = 0.4$ and $\alpha_0 = 0.04$.

### 5.1. Sensitivity analysis of the optimal reinsurance strategy

In this subsection, we focus on the effect of some parameters on the optimal reinsurance strategy. From Eqs. (49) and (51), we know that the robust optimal reinsurance strategy is independent of the parameters in the financial market, which is only related to the parameters in the insurance market, the risk averse and ambiguity averse coefficients of the AAI. Based on the parameters we have set, it follows that $U_1 < U_2$ and $\eta U_2 < (m+\beta_0)U_3$ always hold, so we set $t_2 = T$ and the optimal reinsurance strategy $(q_1^*(t), q_2^*(t)) = (\hat{q}_1(t), \hat{q}_2(t))$. Figs. 1 and 2 present the impact of the parameters $\lambda$ and $\beta_0$ on the optimal reinsurance proportion. We find that the insurer will gradually increase the reinsurance proportion as the time goes on. In Fig. 1, we can see that the reinsurance proportions in both two lines of business of the insurance company decrease with the $\lambda$ increasing, which means that the higher dependence of the two lines of business in an

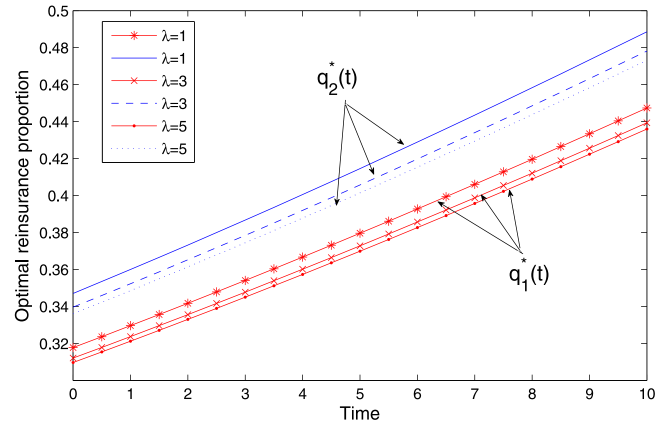


**Fig. 1.** The effect of $\lambda$ on $q_1^*(t)$ and $q_2^*(t)$.

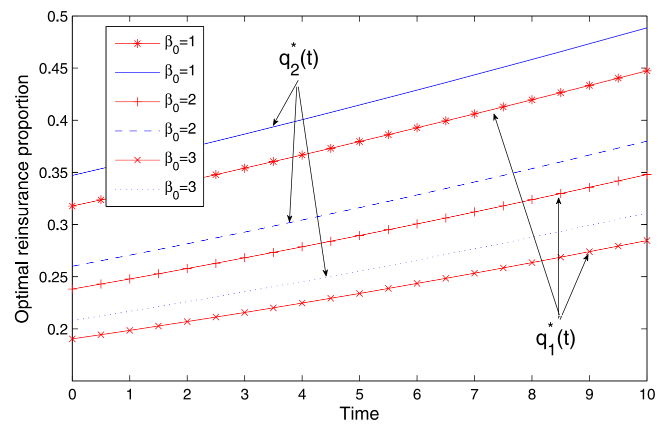


**Fig. 2.** The effect of $\beta_0$ on $q_1^*(t)$ and $q_2^*(t)$.

insurance company, the lower reinsurance retention proportion is arranged by the insurer. An explanation for this phenomenon is that the more dependent degree of the two lines business, the greater potential risks the insurance will bear. So the insurance company disperses risk by purchasing more reinsurance. From Fig. 2, we find that when the insurer has higher ambiguity-averse level $\beta_0$ on the parameters of insurance market, she will arrange lower reinsurance retention proportion.

### 5.2. Sensitivity analysis of the optimal investment strategy

In this subsection, we will investigate the relationship between the robust optimal investment strategy and the model parameters under the CEV model and the Heston's model, respectively. Here we only analyze the strategy at time $t = 0$. Fig. 3 shows the influence of the risk averse coefficient $m$ on the optimal investment strategy. We find that no matter under the CEV model or the Heston's model, the more risk averse the insurer is, the less proportion of the insurance surplus is invested in the risky asset. In addition, under the same risk averse level, compared to the ANI, the AAI invests less proportion of the surplus in the risky asset. This result indicates that the AAI is more conservative due to worrying about the model misspecification.

Fig. 4 presents the effect of $\beta$, ambiguity averse parameter $\beta_1$ and initial price $s_0$ on the optimal investment strategy under the CEV model. From Figs. 4a and b, with the increase of the $\beta_1$, the optimal investment proportion gradually decreases. The explanation for this phenomenon is also because of the misspecification of the model parameter, which causes that the AAI adopts

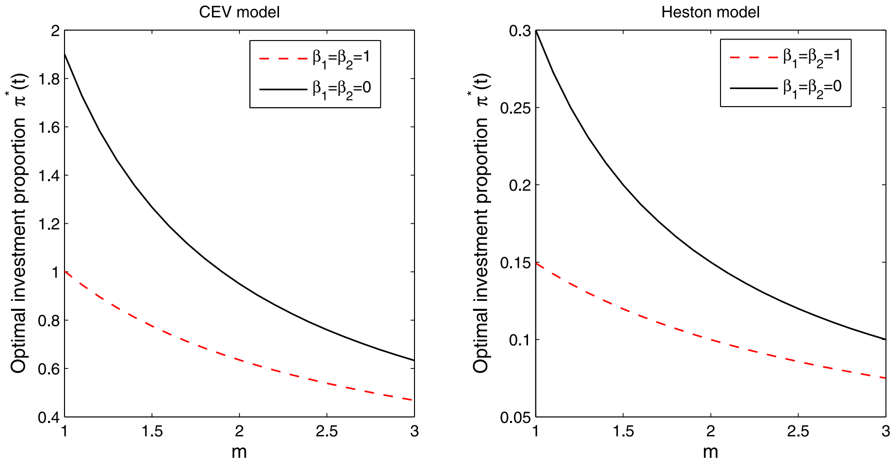


**Fig. 3.** The effect of $m$ on the optimal investment strategy for the CEV and Heston's models.

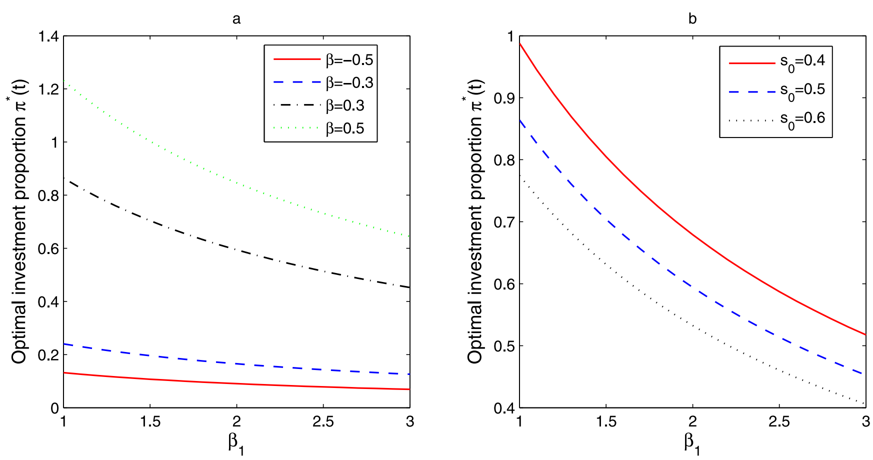


**Fig. 4.** The effect of $\beta$ and $s_0$ on the optimal investment strategy for the CEV model.

more conservative strategy. In Fig. 4a, under the same ambiguity-averse level $\beta_1$, with the increase of $\beta$, the optimal investment proportion will rise as well. For this phenomenon, there is a possible explanation, since at time $t = 0$, we let $s_0 = 0.5$, and the volatility of the risky asset is $\sigma(0) = \varrho s_0^{\beta}$, which is a monotonically decreasing function about $\beta$, and the Sharpe ratio of the risky asset will increase correspondingly. Thus the AAI will invest more surplus in the risky asset to obtain excess returns. However, in Fig. 4b, for a fixed $\beta_1$, when $s_0$ increases, the volatility $\sigma(0)$ correspondingly increases and the Sharpe ratio reduces, the proportion invested in the risky asset falls naturally.

Fig. 5 shows the effects of the parameters $\theta$, $\sigma_0$, $\rho$, $\beta_1$ and $\beta_2$ on the optimal investment strategy under the Heston's model. Figs. 5a and 5b show that with the increase of the ambiguity-averse coefficient $\beta_1$, the corresponding investment proportion decreases. The explanation for this result is similar to that in Fig. 4. From Fig. 5c, we find that the influence of the ambiguity-averse coefficient $\beta_2$ on the optimal investment strategy is not obvious. In Fig. 5a, since the Sharpe ratio of the risky asset would increase with the increase of $\theta$, so its investment proportion will improve naturally. In Fig. 5b, the increase of $\sigma_0$ means that the variance of the volatility becomes larger, thus the AAI will reduce exposure to the risk. Due to the fact that the parameter $\rho$ is the correlation coefficient between $W_1(t)$ and $W_2(t)$, from Fig. 5c, we find that, with stronger positive correlation, the variance of the volatility becomes greater, which leads to that the AAI reduces the risk exposure. On the contrary, with stronger negative correlation, the variance of the volatility becomes smaller. In this case, the AAI is willing to accept more risk and invests more wealth in the risky asset.

### 5.3. Analysis of the utility loss

In this subsection, we will estimate the utility loss due to ignoring the risk of model misspecification. Based on the parameters we have set, it follows that $U_1 < U_2$ and $\eta U_2 \le mU_3$ always hold, so we have $t_2 = \tilde{t}_2 = T$. Thus, the optimal strategy is $(q_1^*(t), q_2^*(t)) = (\hat{q}_1(t), \hat{q}_2(t))$, and the suboptimal strategy is $(\tilde{q}_1^*(t), \tilde{q}_2^*(t)) = (\tilde{q}_1(t), \tilde{q}_2(t))$ for $\forall t \in [0, T]$. Furthermore, using Eq. (71) to calculate the utility loss function from time $t = 0$, we have

$$J(0) = -\frac{1}{m}e^{-rT}\left[(M_1(0) - \hat{M}_1(0)) + (N(0) - \hat{N}(0))\alpha_0\right],$$

where

$$M_1(0) = \int_0^T [-mce^{r(T-s)} + k\phi N(s) + G(\hat{q}_1(s), \hat{q}_2(s))]ds,$$

$$\hat{M}_1(0) = \int_0^T [-mce^{r(T-s)} + k\phi \hat{N}(s) + G(\tilde{q}_1(s), \tilde{q}_2(s))]ds.$$

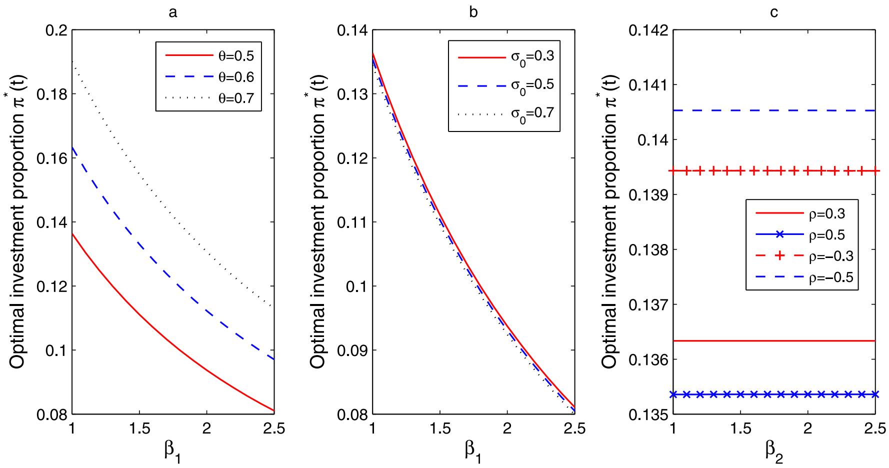


**Fig. 5.** The effect of $\theta$, $\sigma_0$, $\rho$, $\beta_1$ and $\beta_2$ on the optimal investment strategy for the Heston's model.

**Table 2**
Utility loss under the CEV model.

| $T$ | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| Utility loss $J(0)$ | 0.5143 | 1.0189 | 1.5168 | 2.0094 | 2.4937 |

**Table 3**
Utility loss under the Heston's model.

| $T$ | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| Utility loss $J(0)$ | 0.4791 | 0.9543 | 1.4221 | 1.8839 | 2.3430 |

Thus,

$$J(0) = -\frac{1}{m}e^{-rT}\Big[\int_0^T [k\phi(N(s) - \hat{N}(s)) + G(\hat{q}_1(s), \hat{q}_2(s)) - G(\tilde{q}_1(s), \tilde{q}_2(s))]\mathrm{d}s + (N(0) - \hat{N}(0))\alpha_0\Big].$$

Tables 2 and 3 present the value of utility loss $J(0)$ under different investment terminal times $T$ when the volatility of risky asset satisfies the CEV and Heston's models, respectively. In both cases, we find that ignoring the uncertainty risk of the model will lead to significant utility loss. In addition, with increase of the investment time horizon $T$, the value of utility loss will increase as well.

## 6. Conclusion

In this paper, we investigate an optimal investment–reinsurance problem for an insurer with both risk averse and ambiguity averse, and assume that the insurance company has multiple dependent risks. Using the robust optimal control method, we obtain the optimal investment and reinsurance strategies and the corresponding value function. Lastly, some numerical analyses are presented. Based on the theoretical and numerical analyses, we find that: (i) The robust optimal reinsurance strategy depends on the time, interest rate, ambiguity-aversion coefficient and the parameters related to the insurance market, the robust optimal investment strategy only depends on interest rate, ambiguity-aversion coefficient and the parameters related to the financial market. (ii) Both the risk aversion and ambiguity aversion levels of the AAI have significant influence on the optimal strategy. (iii) With higher dependence degree on the two lines of business, the retention levels of the reinsurance will become smaller. (iv) Under the assumption of stochastic volatility, no matter it satisfies the CEV model or the Heston's model, ignoring the risk of model uncertainty will lead to significant utility loss, and the longer the investment time horizon is, the higher the value of utility loss is.

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## Appendix. Proof of Theorem 3.1

Since $H(t, x, \alpha) \in C^{1,2,2}$ for any $(t, x, \alpha) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+$, based on the Corollary 1.2 of Kraft (2004), we can prove Theorem 3.1 if $u^*$ and the corresponding candidate value function $H(t, x, \alpha)$ satisfy the following properties:

(1) $u^*$ is an admissible strategy;

(2) $\mathbb{E}^{\mathbb{Q}^*}(\sup_{t\in[0,T]} \mid H(t, X^{u^*}(t), \alpha(t)) \mid^4) < \infty$, where $\mathbb{Q}^*$ is defined by $\zeta^*(t)$ in Eq. (13) with $\gamma^*(t, \alpha(t))$;

(3) $\mathbb{E}^{\mathbb{Q}^*}\left(\sup_{t\in[0,T]} \left| \frac{(\gamma_0^*(t))^2}{2\psi_0(t)} + \frac{(\gamma_1^*(t,\alpha(t)))^2}{2\psi_1(t)} + \frac{(\gamma_2^*(t,\alpha(t)))^2}{2\psi_2(t)} \right|^2\right) < \infty.$

We will verify above properties one by one. To make the proof process to be understood easily and logically, we firstly prove property (2) and then prove property (1).

**Proof of property (2).** Substituting $\pi^*(t, \alpha(t))$ and $q^*(t) = (q_1^*(t), q_2^*(t))$ into Eq. (16) (here we let $z = 2$), we have

$$\begin{aligned} X^{u^*}(t) = x_0e^{rt} &+ \int_0^t e^{r(t-s)}\big[Q_1(q^*(s)) - \beta_0 e^{r(T-s)}Q_2(q^*(s)) \\ &+ \frac{m}{m+\beta_1}\theta^2\alpha(s)e^{-r(T-s)}(\frac{1}{m+\beta_1} + \frac{k_1}{m\theta}N(s))\big]\mathrm{d}s \\ &+ \int_0^t e^{r(t-s)}\sqrt{Q_2(q^*(s))}\mathrm{d}W_0^{\mathbb{Q}^*}(s) \\ &+ \int_0^t e^{-r(T-t)}\theta\sqrt{\alpha(s)}(\frac{1}{m+\beta_1} + \frac{k_1}{m\theta}N(s))\mathrm{d}W_1^{\mathbb{Q}^*}(s), \quad (72) \end{aligned}$$

where $Q_1(q^*(s)) = c - \delta(q^*(s)) - \sum_{l=1}^2 a_l q_l^*(s)$, $Q_2(q^*(s)) = \sum_{l=1}^2 \sigma_l^2(q_l^*(s))^2 + 2\lambda q_1^*(s)q_2^*(s)\mu_1\mu_2$. Inserting (72) into the candidate value function, we have

$$\begin{aligned} &|H(t, X^{u^*}(t), \alpha(t))^4| \\ &= \frac{1}{m^4}\exp\{-4mX^{u^*}(t)e^{r(T-t)} + 4M(t) + 4N(t)\alpha(t)\} \\ &\le K_1 \exp\{-4mX^{u^*}(t)e^{r(T-t)}\} \\ &\le K_2 \exp\Big\{-4m\int_0^t \frac{m}{m+\beta_1}\theta^2\alpha(s)[\frac{1}{m+\beta_1} + \frac{k_1}{m\theta}N(s)]\mathrm{d}s \end{aligned}$$

$$\exp\{-4m\int_0^t \frac{m}{m+\beta_1}\theta^2\alpha(s)[\frac{1}{m+\beta_1}+\frac{k_1}{m\theta}N(s)]ds - 4m\int_0^t \theta\sqrt{\alpha(s)}[\frac{1}{m+\beta_1}+\frac{k_1}{m\theta}N(s)]dW_1^{\mathbb{Q}^*}(s)\}$$
$$= \underbrace{\exp\{-\int_0^t 16m^2\theta^2\alpha(s)[\frac{1}{m+\beta_1}+\frac{k_1}{m\theta}N(s)]^2ds - \int_0^t 4m\theta\sqrt{\alpha(s)}[\frac{1}{m+\beta_1}+\frac{k_1}{m\theta}N(s)]dW_1^{\mathbb{Q}^*}(s)\}}_{F}$$
$$\cdot \underbrace{\exp\{\int_0^t \Big(16m^2\theta^2[\frac{1}{m+\beta_1}+\frac{k_1}{m\theta}N(s)]^2 - 4m^2\frac{\theta^2}{m+\beta_1}[\frac{1}{m+\beta_1}+\frac{k_1}{m\theta}N(s)]\Big)\alpha(s)ds\}}_{Z}. \quad (75)$$

**Box I.**

$$- 4m\int_0^t e^{r(T-s)}\sqrt{Q_2(q^*(s))}dW_0^{\mathbb{Q}^*}(s)$$
$$- 4m\int_0^t \theta\sqrt{\alpha(s)}[\frac{1}{m+\beta_1}+\frac{k_1}{m\theta}N(s)]dW_1^{\mathbb{Q}^*}(s)\Big\}, \quad (73)$$

where $K_1$ and $K_2$ are two constants and satisfy

$$K_1 \geq \frac{1}{m^4}e^{4M(t)+4N(t)\alpha(t)}, \quad \text{for } t\in[0,T],\ \mathbb{P}-a.s.$$
$$K_2 \geq K_1 e^{-4m[x_0e^{rT}+\int_0^t e^{r(T-s)}(Q_1(q^*(s))-\beta_0e^{r(T-s)}Q_2(q^*(s)))ds]},$$

for $t\in[0,T]$.

Note that $m\sqrt{Q_2(q^*(s))}e^{r(T-s)}$ is bounded on $s\in[0,T]$, we find that the exponential integral

$$\exp\{\int_0^t -4me^{r(T-s)}\sqrt{Q_2(q^*(s))}dW_0^{\mathbb{Q}^*}(s)\}$$
$$= \exp\{\int_0^t 8m^2e^{2r(T-s)}Q_2(q^*(s))ds\}$$
$$\cdot \underbrace{\exp\{-\int_0^t 8m^2e^{2r(T-s)}Q_2(q^*(s))ds + \int_0^t -4me^{r(T-s)}\sqrt{Q_2(q^*(s))}dW_0^{\mathbb{Q}^*}(s)\}}_{\text{martingale}}.$$

Consequently, it follows that

$$\mathbb{E}^{\mathbb{Q}^*}\Big(\exp\{\int_0^t -4me^{r(T-s)}\sqrt{Q_2(q^*(s))}dW_0^{\mathbb{Q}^*}(s)\}\Big) < \infty. \quad (74)$$

Then, we aim to find an estimator for

$$\exp\{-4m\int_0^t \frac{m}{m+\beta_1}\theta^2\alpha(s)[\frac{1}{m+\beta_1}+\frac{k_1}{m\theta}N(s)]ds - 4m$$
$$\times\int_0^t \theta\sqrt{\alpha(s)}[\frac{1}{m+\beta_1}+\frac{k_1}{m\theta}N(s)]dW_1^{\mathbb{Q}^*}(s)\}.$$

Notethat, Eq. (75) is given in Box I. For the term $F$,

$$\mathbb{E}^{\mathbb{Q}^*}(F^2) = E^{\mathbb{Q}^*}\Big[\exp\{-\int_0^t 32m^2\theta^2\alpha(s)[\frac{1}{m+\beta_1}+\frac{k_1}{m\theta}N(s)]^2ds$$
$$-\int_0^t 8m\theta\sqrt{\alpha(s)}[\frac{1}{m+\beta_1}+\frac{k_1}{m\theta}N(s)]dW_1^{\mathbb{Q}}(s)\}\Big]$$
$$<\infty, \quad (76)$$

since $F^2$ is a supermartingale. Due to that $-8m\theta[\frac{1}{m+\beta_1}+\frac{k_1}{m\theta}N(s)]$ is bounded on $[0,T]$, according to Lemma 4.3 in Taksar and Zeng (2009), $F^2$ is a martingale.

For the term $Z$, we have

$$\mathbb{E}^{\mathbb{Q}^*}[Z^2] = \mathbb{E}^{\mathbb{Q}^*}\Big[\exp\{\int_0^t \Big(32m^2\theta^2[\frac{1}{m+\beta_1}+\frac{k_1}{m\theta}N(s)]^2$$
$$- 8m^2\frac{\theta^2}{m+\beta_1}[\frac{1}{m+\beta_1}+\frac{k_1}{m\theta}N(s)]\Big)\alpha(s)ds\}\Big].$$

Based on the Theorem 5.1 in Taksar and Zeng (2009), the sufficient condition for $\mathbb{E}^{\mathbb{Q}^*}(Z^2)<\infty$ is as follows:

$$32m^2\theta^2[\frac{1}{m+\beta_1}+\frac{k_1}{m\theta}N(t)]^2 - 8m^2\frac{\theta^2}{m+\beta_1}[\frac{1}{m+\beta_1}$$
$$+\frac{k_1}{m\theta}N(t)] \leq \frac{k^2}{2(k_1^2+k_2^2)}. \quad (77)$$

For $\forall t\in[0,T]$, notice that $-\frac{b_3}{b_1+b_2}<N(t)<0$, by the property of quadratic function, when the technical condition (48) is satisfied, the inequality (77) also holds. Applying (73), we obtain

$$\mathbb{E}^{\mathbb{Q}^*}\Big[\Big|H(t,X^{u^*}(t),\alpha(t))\Big|^4\Big]$$
$$\leq K_2\mathbb{E}^{\mathbb{Q}^*}\Big[\exp\{\int_0^t -4me^{r(T-s)}\sqrt{Q_2(q^*(s))}dW_0^{\mathbb{Q}^*}(s)\}\Big]\mathbb{E}^{\mathbb{Q}^*}[F\cdot Z]$$
$$\leq K_3\mathbb{E}^{\mathbb{Q}^*}[F\cdot Z] \leq K_3(\mathbb{E}^{\mathbb{Q}^*}[F^2]\mathbb{E}^{\mathbb{Q}^*}[Z^2])^{1/2} < \infty. \quad (78)$$

In the above estimation, the first inequality follows from (73) and due to the fact that $W_0^{\mathbb{Q}^*}(t)$ is independent of $W_1^{\mathbb{Q}^*}(t)$ and $W_2^{\mathbb{Q}^*}(t)$. The second inequality follows from (74) and due to the following inequality

$$K_3 \geq K_2\mathbb{E}^{\mathbb{Q}^*}\Big[\exp\{\int_0^t -4me^{r(T-s)}\sqrt{Q_2(q^*(s))}dW_0^{\mathbb{Q}^*}(s)\}\Big].$$

The third inequality follows from the Cauchy–Schwarz inequality. And the last inequality is from Eq. (76) and $\mathbb{E}^{\mathbb{Q}^*}[Z^2]<\infty$. Therefore, property (2) is proved.

**Proof of property (1).** From the process of solving HJB equation, we know the optimal strategy $u^*(t,\alpha(t))$ is progressively measurable. From Eqs. (25), (49) and (51), $q_1^*(t)$, $q_2^*(t)$ are deterministic and state independent. The optimal strategy $\pi^*(t,\alpha(t))$ depends on the state process $\alpha(t)$, $\alpha(t)$ is a mean-reverting square root process, although it is generally an unbounded random variable for any fixed given time, its first and second order moments are bounded (the detailed proof for the boundedness of its moments can be found in Mao (1997, p. 308) or Kwok (1998, p. 397)), thus condition (i) in Definition 2.1 holds. For the condition (ii) of Definition 2.1, by the proof of property (2), we have deduced that the solution of Eq. (16) has the form of Eq. (72). On the other hand, in property (2), we have proved $\mathbb{E}^{\mathbb{Q}^*}\Big[\Big|H(t,X^{u^*}(t),\alpha(t))\Big|^4\Big]<+\infty$ for $t\in[0,T]$. Notice that $U(X^{u^*}(T)) = H(T,X^{u^*}(T),\alpha(T))$, by the similar method used in the proof of property (2), we can prove $\mathbb{E}^{\mathbb{Q}^*}\Big[\Big|H(T,X^{u^*}(T),\alpha(T))\Big|\Big]<+\infty$. Thus, $u^*$ is an admissible strategy.

**Proof of property (3).** Let $\Gamma(t) = \frac{(\gamma_0^*(t))^2 m}{2\beta_0} + \frac{(\gamma_1^*(t,\alpha(t)))^2 m}{2\beta_1} + \frac{(\gamma_2^*(t,\alpha(t)))^2 m}{2\beta_2}$. Since for $\forall l > 0$, $\mathbb{E}^{\mathbb{Q}^*}[\alpha(t)^l] < \infty$, by Eq. (53), we have $\mathbb{E}^{\mathbb{Q}^*}[\Gamma(t)^4] < \infty$. From Eq. (19), we notice that $H(t,x,\alpha) = \frac{\beta_0}{-m\psi_0} = \frac{\beta_1}{-m\psi_1} = \frac{\beta_2}{-m\psi_2}$. So we have

$$\mathbb{E}^{\mathbb{Q}^*}\left(\sup_{t\in[0,T]}\left|\frac{(\gamma_0^*(t))^2}{2\psi_0(t)} + \frac{(\gamma_1^*(t,\alpha(t)))^2}{2\psi_1(t)} + \frac{(\gamma_2^*(t,\alpha(t)))^2}{2\psi_2(t)}\right|^2\right)$$
$$= \mathbb{E}^{\mathbb{Q}^*}\left(\sup_{t\in[0,T]}\left|\Gamma(t)H(t,X^{u^*}(t),\alpha(t))\right|^2\right)$$
$$= \mathbb{E}^{\mathbb{Q}^*}\left(\sup_{t\in[0,T]}|\Gamma(t)|^2|H(t,X^{u^*}(t),\alpha(t))|^2\right)$$
$$\le \left[\mathbb{E}^{\mathbb{Q}^*}\left(\sup_{t\in[0,T]}|\Gamma(t)|^4\right)\right]^{\frac{1}{2}}$$
$$\times\left[\mathbb{E}^{\mathbb{Q}^*}\left(\sup_{t\in[0,T]}|H(t,X^{u^*}(t),\alpha(t))|^4\right)\right]^{\frac{1}{2}} < \infty.$$

The first inequality above follows from the Cauchy–Schwarz inequality and the second inequality follows from $\mathbb{E}^{\mathbb{Q}^*}[\Gamma(t)^4] < \infty$ and (78).

Now all properties are proven, using Corollary 1.2 in Kraft (2004), we can guarantee that $u^*$ is an optimal strategy and $H(t,x,\alpha)$ is the corresponding value function, i.e., $V(t,x,\alpha) = H(t,x,\alpha)$.

## References

Anderson, E.W., Hansen, L.P., Sargent, T.J., 1999. Robustness, detection and the price of risk. Working paper. University of Chicago. https://fles.nyu.edu/ts43/public/research/.svn/textbase/ahs3.pdf.snv-base.

Anderson, E.W., Hansen, L.P., Sargent, T.J., 2003. A quartet of semi-groups for model specifcation, robustness, prices of risk, and model detection. J. Eur. Econom. Assoc. 1 (1), 68–123.

Bai, L.H., Cai, J., Zhou, M., 2013. Optimal reinsurance policies for an insurer with a bivariate reserve risk process in a dynamic setting. Insurance Math. Econom. 53 (3), 664–670.

Bi, J.N., Liang, Z.B., Xu, F.J., 2016. Optimal mean–variance investment and reinsurance problems for the risk model with common shock dependence. Insurance Math. Econom. 70, 245–258.

Branger, N., Larsen, L.S., 2013. Robust portfolio choice with uncertainty about jump and diffusion risk. J. Bank. Financ. 37 (12), 5036–5047.

Cao, Y.S., Wan, N.Q., 2009. Optimal proportional reinsurance and investment based on Hamilton–Jacobi-Bellman equation. Insurance Math. Econom. 45, 157–162.

Grandell, J., 1991. Aspects of Risk Theory. Springer-Verlag, New York.

Gu, A.L., Viens, F.G., Yao, H.X., 2018. Optimal robust reinsurance-investment strategies for insurers with mean reversion and mispricing. Insurance Math. Econom. 80, 93–109.

Karatzas, I., Shreve, S., 2012. Brownian Motion and Stochastic Caculus. Springer Science & Business Media.

Kraft, H., 2004. Optimal portfolios with stochastic interest rates. Lecture Notes Econom. Math. Syst..

Kwok, Y.K., 1998. Mathematical Models of Financial Derivatives. Springer, Berlin.

Larsen, L.S., Munk, C., 2012. The costs of suboptimal dynamic asset allocation: General results and applications to interest rate risk, stock volatility risk, and growth/value tilts. J. Econom. Dynam. Control 36 (2), 266–293.

Li, D.P., Rong, X.M., Zhao, H., 2017. Equilibrium excess-of-loss reinsurance-investment strategy for a mean–variance insurer under stochastic volatility model. Comm. Statist. Theory Methods 46 (19), 9459–9475.

Li, D.P., Zeng, Y., Yang, H.L., 2018. Robust optimal excess-of-loss reinsurance and investment strategy for an insurer in a model with jumps. Scand. Actuar. J. 2018 (2), 145–171.

Liang, Z.B., Guo, J.Y., 2011. Optimal combining quota-share and excess of loss reinsurance to maximize the expected utility. J. Appl. Math. Comput. 36, 11–25.

Liang, Z.B., Yuen, K.C., 2016. Optimal dynamic reinsurance with dependent risks: variance premium principle. Scand. Actuar. J. 2016 (1), 18–36.

Maenhout, P.J., 2004. Robust portfolio rules and asset pricing. Rev. Financ. Stud. 17 (4), 951–983.

Mao, X., 1997. Stochastic Differential Equations and Applications, Horwood.

Ming, Z.Q., Liang, Z.B., Zhang, C.B., 2016. Optimal mean–variance reinsurance with common shock dependence. ANZIAM J. 58 (2), 162–181.

Pun, C.S., 2018. Robust time-inconsisitent stochastic control problems. Automatica 94, 249–257.

Pun, C.S., Wong, H.Y., 2015. Robust investment-reinsurance optimization with multiscale stochastic volatility. Insurance Math. Econom. 62, 245–256.

Shen, Y., Zeng, Y., 2015. Optimal investment-reinsurance strategy for mean–variance insurers with square-root factor process. Insurance Math. Econom. 62, 118–137.

Taksar, M., Zeng, X.D., 2009. A general stochastic volatility model and optimal portfolio with explicit solutions. Working Paper.

Xu, L., Zhang, L.M., Yao, D.J., 2017. Optimal investment and reinsurance for an insurer under Markov-modulated financial market. Insurance Math. Econom. 74, 7–19.

Yi, B., Li, Z.F, Viens, F.G., Zeng, Y., 2013. Robust optimal control for an insurer with reinsurance and investment under heston's stochastic volatility model. Insurance Math. Econom. 53 (3), 601–614.

Yi, B., Viens, F.G., Li, Z.F., Zeng, Y., 2015. Robust optimal strategies for an insurer with reinsurance and investment under benchmark and mean–variance criteria. Scand. Actuar. J. 2015 (8), 725–751.

Yuen, K.C., Liang, Z.B., Zhou, M., 2015. Optimal proportional reinsurance with common shock dependence. Insurance Math. Econom. 64, 1–13.

Zeng, Y., Li, D.P., Gu, A.L., 2016. Robust equilibrium reinsurance-investment strategy for a mean–variance insurer in a model with jumps. Insurance Math. Econom. 66, 138–152.

Zhang, N., Jin, Z., Li, S.M., Chen, P., 2016b. Optimal reinsurance under dynamic VaR constraint. Insurance Math. Econom. 71, 232–243.

Zhang, X., Meng, H., Zeng, Y., 2016a. Optimal investment and reinsurance strategies for insurers with generalized mean–variance premium principle and no-short selling. Insurance Math. Econom. 67, 125–132.

Zheng, X.X., Zhou, J.M., Sun, Z.Y., 2016. Robust optimal portfolio and proportional reinsurance for an insurer under a CEV model. Insurance Math. Econom. 67, 77–87.