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Pre-commitment and equilibrium investment strategies for the DC pension plan with regime switching and a return of premiums clause

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A B S T R A C T

This paper studies an optimal investment problem for a defined-contribution (DC) pension plan during the accumulation phase, where a pension member contributes a predetermined amount of money as a premium and then the manager of the pension fund invests the premium in a financial market to increase the value of the accumulation. To protect the rights of pension members who die before retirement, a return of premiums clause is introduced, under which a member who dies before retirement can withdraw all the premiums she has contributed. We assume that the financial market consists of one risk-free asset and multiple risky assets, the returns of the risky assets depend on the market states, the evolution of the market states is described by a Markov chain, and the transition matrixes are time-varying. The pension fund manager aims to maximize the expected terminal wealth of each surviving member at retirement and to minimize the risk measured by the variance of her terminal wealth, which are two conflicting objectives. We formulate the investment problem as a discrete-time mean–variance model. Since the model is time-inconsistent, we seek its pre-commitment and equilibrium strategies. Using the embedding technique and the dynamic programming method, we obtain the pre-commitment strategy and the corresponding efficient frontier in closed form. Applying the game theory and the extended Bellman equation, we derive the analytical expressions of the equilibrium strategy and the corresponding efficient frontier. For the two obtained investment strategies and their corresponding efficient frontiers, as well as the impact of regime switching and the return of premiums clause on them, some interesting theoretical and numerical results are found.

1. Introduction

Recently, the investment management of pension funds has become increasingly important due to population aging. There are two basic types of pension plans: the defined contribution (DC) pension plan and the defined benefit (DB) pension plan. In a DB pension plan, the benefits are defined in advance by the sponsor. In a DC pension plan, contributions are fixed and benefits depend solely on the investment returns of the plan. Compared with the DB pension plan, the DC pension plan has an advantage to ease the pressure of the social security system by transferring investment risk and longevity risk from sponsors to the pension plan members. Therefore, a growing number of countries have partially or even completely shifted from the DB pension plan to the DC pension plan. Consequently, the asset allocation for DC pension plans, which is also the topic of this paper, has attracted much attention in recent years.

The mean–variance formulation, which was proposed by Markowitz (1952), has become an important criterion to study the asset allocation for DC pension plans, especially after the breakthrough of solving the dynamic mean–variance model for multi-period and continuous-time cases by Li and Ng (2000) and Zhou and Li (2000). However, all of the dynamic investment problems under the mean–variance criterion are time-inconsistent, because the non-separability of the variance operator leads to a failure of the Bellman optimality principle. Fortunately, Strotz (1955) proposed two ways to deal with the time-inconsistent problems. The first one is to fix one initial point, then try to find the optimal strategy that maximizes the objective function without regard to whether the latter points are optimal or not. This is called a pre-commitment strategy, which is a global optimal strategy but a time-inconsistent strategy. The second one is to tackle the time inconsistency seriously by using the game theory approach to obtain an equilibrium strategy, which is a time-consistent strategy

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but not a global optimal strategy. Björk and Murgoci (2010, 2014) and Björk et al. (2017) made a detailed study of this theory on discrete and continuous-time cases.

Several scholars studied the pre-commitment investment strategies for DC pension plans. For example, Menoncin and Vigna (2013) studied a mean–variance target-based optimization problem for the DC pension plan with stochastic interest rate. Guan and Liang (2015) considered the mean–variance efficiency of the DC pension plan with stochastic interest rate and mean-reverting returns. Nkeki (2013) studied the mean–variance portfolio for the DC pension plan with stochastic salary and compared it with the optimal portfolios under the quadratic utility function, the power utility function and the exponential utility function. Other contributions to this problem include among others, Højgaard and Vigna (2007), Vigna (2014) and Yao et al. (2016a). Researches on the equilibrium investment strategy for the DC pension plans are very few. Wu et al. (2015) studied an equilibrium investment strategy for the DC pension plan with inflation risk and salary risk. Meanwhile, they compared their strategy with the pre-commitment investment strategy and obtained some distinct properties of the two kinds of strategies. Li et al. (2016) investigated an equilibrium investment strategy for the DC pension plan with stochastic salary under CEV model. He and Liang (2013) and Wu and Zeng (2015) also considered the equilibrium investment strategy for the DC pension plan. In addition, some scholars studied both of the two strategies for DC pension plans simultaneously. For instance, Sun et al. (2016) studied both the pre-commitment and equilibrium strategies for a DC pension plan under a jump–diffusion model and obtained several different characteristics of them.

In the afore-mentioned papers, the returns of the risky assets are assumed to be independent of the economy state. However, many investment practice and empirical studies demonstrate that some macroeconomic variables, such as the exchange rate, the inflation rate, the interest rate and the GDP growth rate, have significant impacts on the return and volatility of risky assets, see Asprem (1989) and Engle et al. (2008). Hence, the economy states should be considered in the research of the DC pension fund investment management. Markov regime switching model, which was originally proposed by Hamilton (1989), has been proven to be a good way to describe the stochastic evolution of the financial market states. The model uses a discrete-time or continuous-time finite state Markov chain to describe the states (regimes) of an economy. In the previous literature, some scholars investigated the mean–variance portfolio problems with regime switching, such as Zhou and Yin (2003), Çakmak and Özekici (2006), Chen et al. (2014), Wu and Chen (2015) and Chen et al. (2016). Some scholars studied the asset–liability management problems in Markov regime switching market, such as Chen and Yang (2011) and Yao et al. (2016b). Some scholars considered the investment–consumption problems with environment uncertainty, see, e.g., Li et al. (2008) and Gassiat et al. (2014). However, there are only a few studies related to the investment management of the DC pension fund with regime-switching. Korn et al. (2011) and Chen and Delong (2015) considered the continuous-time asset allocation for DC pension funds with regime switching. Yao et al. (2016a) studied the multi-period investment management for the DC pension fund with regime switching and mortality risk, and obtained a pre-commitment investment strategy without considering the time-consistent investment strategy. To the best of our knowledge, the time-consistent investment strategy for the DC pension plan with regime switching under a multi-period mean–variance framework has not been studied yet.

In a DC pension plan, since the members may die during the accumulation process, it is natural that the mortality risk should be considered in the investment management of DC pension funds. Recent contributions to the study of DC pension plans in the presence of mortality risk can be referred to Yao et al. (2014, 2016a) and Wu and Zeng (2015), in which the mortality risk is considered from the perspective of the pension plan members. In reality, however, many DC pension plans are entrusted to specialized management agencies, such as the Chinese enterprise pension fund. Hence, the pension fund manager should also consider the mortality risk. In order to protect the rights of the members who die before retirement, most of DC pension plans have return of premium clauses. In this kind of actuarial clauses, the dead members can withdraw all of the premiums that they have contributed or the premiums that have been accumulated according to a predetermined interest rate. He and Liang (2013) are the first to incorporate return of premiums clauses into asset allocation of DC pension plans. Li et al. (2017), Sun et al. (2016) and Sheng and Rong (2014) also considered return of premiums clauses for DC pension plans. However, all of these papers are within the continuous-time framework. To the best of our knowledge, the multi-period mean–variance investment problem for the DC pension plan with a return of premiums clause has not been studied before.

With the above in mind, the purpose of this work is to study both the pre-commitment and equilibrium strategies for a DC pension plan with regime switching and a return of premiums clause under a multi-period mean–variance framework. Utilizing the embedding technique and the dynamic programming method, we obtain the pre-commitment strategy and the corresponding efficient frontier in closed-form. Applying the game theory and the extended Bellman equation, we derive analytical expressions for the equilibrium strategy and the corresponding efficient frontier. Moreover, numerical analysis is conducted to compare the two strategies and the two efficient frontiers and analyze the impact of the regime switching and the return of premiums clause on them as well.

The main contributions of this paper are as follows. (i) We consider the multi-period mean–variance investment problem for a DC pension plan with both regime switching and a return of premiums clause, which was not considered in the afore-mentioned literature. (ii) In the accumulation phase of the DC pension plan, different from that in Yao et al. (2014, 2016a) and Wu and Zeng (2015), the mortality risk is considered from the perspective of the pension fund manager. (iii) We obtain both the pre-commitment and equilibrium strategies for our problem in closed form. (iv) The approach of our derivation is rather technical and may shed light on the research of the relevant dynamic optimization problems.

The remaining of this paper is organized as follows. We model the multi-period mean–variance investment problem for the DC pension plan with regime switching and a return of premiums clause in Section 2. In Sections 3 and 4, we derive the pre-commitment strategy and the equilibrium strategy as well as their corresponding efficient frontiers respectively. In Section 5, we briefly discuss two special cases of our model. Numerical analysis with real data is conducted in Section 6. We finish with a conclusion in Section 7.

2. Problem formulation

We are concerned with a DC pension plan in which the accumulation process of a member starts from year 0 or age y and ends in the year of retirement T or age $y+T$. During the accumulation phase, as long as the member is alive, she needs to contribute a predetermined amount of money as a premium at the beginning of every year. We assume that the premium in year k is C_k.

To protect the rights of pension members who die before retirement, we introduce the return of premiums clause: the dead members can withdraw all of the premiums that they have contributed. That is, when a member dies in time interval $(k, k+1]$ for $k = 0, 1, \ldots, T-1$, all of the premiums $\sum_{l=0}^{k} C_l$ that she has contributed will be returned to her at time $k+1$, but the appreciation on the investments of the accumulation (i.e., the difference between the return and the accumulation) will be equally distributed to the surviving members. Denote by q_{k+y} the probability that a person will die between ages $k+y$ and $k+y+1$ given that she is alive at age $k+y$, and $p_{k+y}=1-q_{k+y}$ be the probability that a person survives to age $k+y+1$ given that she is alive at age $k+y$.

2.1. Financial market

Suppose that a financial market under consideration has finite states (regimes) and switches randomly among them. Let $\Pi=\{1,2,\ldots,J\}$ be the state set and ξ_k be the state at time k $(k=0,1,\ldots,T-1)$. We assume that the state process $\{\xi_k, k=0,1,\ldots,T-1\}$ follows a Markov chain with time-dependent transition matrix $Q(k)=(q_{ij}(k))_{J\times J}$, where $q_{ij}(k)=\Pr(\xi_{k+1}=j|\xi_k=i)$ is the transition probability from state $\xi_k=i$ at time k to state $\xi_{k+1}=j$ at time $k+1$ $(i,j\in\Pi)$.

The pension manager will invest the premiums in the financial market to increase the value of the accumulation. Assume that the financial market consists of one risk-free asset and n risky assets. For $k=0,1,\ldots,T-1$, let $r_k(>0)$ be the return of the risk-free asset in year k and $S_k^l(\xi_k)$ the excess return of risky asset l $(l=1,2,\ldots,n)$ over the risk-free asset in year k under a given state ξ_k. Then the return of risky asset l in year k is $r_k+S_k^l(\xi_k)$. For $k=0,1,\ldots,T-1$, $m,l=1,2,\ldots,n$ and $i\in\Pi$, let $S_k(i)=\left(S_k^1(i),S_k^2(i),\ldots,S_k^n(i)\right)'$ be the excess return vector, $s_k(i)=\mathbb{E}[S_k(i)]$ the expected excess return vector, and $\mathbf{cov}_k(i)=\left(\sigma_k^{m,l}(i)\right)_{n\times n}$ the covariance matrix, where $\sigma_k^{m,l}(i)=\operatorname{cov}\left(S_k^m(i),S_k^l(i)\right)$ is the covariance between $S_k^m(i)$ and $S_k^l(i)$. Throughout the paper, we denote the transpose of a matrix A by A'. Similar to most of the existing literature, we make the following assumptions throughout this paper.

Assumption 2.1. For $k,m=0,1,\ldots,T-1$, random vectors $S_k(i)$ and $S_m(j)$ are statistically independent for any $i,j\in\Pi$ when $k\neq m$.

Assumption 2.2. $\mathbf{cov}_k(i)$ is positive definite for all $k=0,1,\ldots,T-1$ and all $i\in\Pi$.

Assumption 2.3. For all $k=0,1,\ldots,T-1$ and all $i\in\Pi$, $s_k(i)\neq\vec{0}_n$, where $\vec{0}_n$ is the n-dimensional zero vector.

Assumption 2.4. Transaction cost and tax are not considered and short-selling is allowed.

2.2. Wealth process and optimization problem

We now give the wealth process of the pension member according to the actuarial rules. For $k=0,1,\ldots,T-1$, let $\pi_k(\xi_k)=\left(\pi_k^1(\xi_k),\pi_k^2(\xi_k),\ldots,\pi_k^n(\xi_k)\right)'$ be the amount invested in the n risky assets at time k in state ξ_k, and $\pi(k):=\{\pi_j(\xi_j),\ j=k,k+1,\ldots,T-1\}$ represent the strategy throughout years $k,k+1,\ldots,T-1$. Denote by X_k^π the wealth of the member under strategy π at time k, incorporating the contribution C_k at time k, then the amount invested in the risk-free asset at time k is $X_k^\pi+C_k-\sum_{l=1}^n\pi_k^l(\xi_k)$. According to the return of premiums clause, if a pension member dies in time interval $(k,k+1]$, then all the premiums $\sum_{l=0}^k C_l$ contributed by the member will be returned to her at time $k+1$; if the member is still alive at time $k+1$, the manager will distribute to the member the difference between the return and the accumulation from the members who die during the time interval $(k,k+1]$,

$$F_{k+1}=\frac{q_{k+y}\left[\left(X_k^\pi+C_k-\sum_{l=1}^n\pi_k^l(\xi_k)\right)r_k+\sum_{l=1}^n\pi_k^l(\xi_k)\left(r_k+S_k^l(\xi_k)\right)-\beta\sum_{l=0}^k C_l\right]}{p_{k+y}},$$

which is an actuarial value, where β is a parameter with values 0 and 1. Obviously, when $\beta=0$, the return of premiums clause is not considered, that is to say, a pension plan member will gain nothing if she dies during the accumulation phase. While when $\beta=1$, the pension plan member can get all of the premiums she has ever contributed when she dies.

Hence, the wealth of the surviving member at time $k+1$ is

$$\begin{aligned}X_{k+1}^\pi&=\left(X_k^\pi+C_k-\sum_{l=1}^n\pi_k^l(\xi_k)\right)r_k+\sum_{l=1}^n\pi_k^l(\xi_k)\left(r_k+S_k^l(\xi_k)\right)+F_{k+1}\\&=\frac{\left(X_k^\pi+C_k\right)r_k+S_k'(\xi_k)\pi_k(\xi_k)-\beta q_{k+y}\sum_{l=0}^k C_l}{p_{k+y}}\\&=A_{k,k}X_k^\pi+B_k(\beta)+\frac{S_k'(\xi_k)\pi_k(\xi_k)}{p_{k+y}},\end{aligned}\tag{1}$$

where $A_{k,m}=\prod_{l=k}^m\frac{r_l}{p_{l+y}}>0$ $(m=k,k+1,\ldots,T-1)$, $B_k(\beta)=\frac{C_kr_k-\beta q_{k+y}\sum_{l=0}^k C_l}{p_{k+y}}$.

Let $\wp_k$ be the family of filters, denoting the information available to the pension manager up to time k, i.e., $\wp_k:=\sigma\{(X_s^\pi,\xi_s)|0\le s\le k\}$, which is a σ-field. $\pi(k)=\{\pi_j(\xi_j),\ j=k,k+1,\ldots,T-1\}$, which is an investment strategy starting from time k, is called time-k admissible if $\pi_j(\xi_j)$ is adapted to $\wp_j$ for all $j=k,k+1,\ldots,T-1$. Denote by Θ_k the collection of all time-k admissible investment strategies.

The pension manager aims to maximize the expected terminal wealth of each surviving member at the time of retirement and to minimize the risk measured by the variance of her terminal wealth, which are two conflicting objectives. Therefore, we formulate the investment problem as the following mean–variance model:

$$\max_\pi\{\mathbb{E}\left(X_T^\pi\right)-\omega\operatorname{Var}\left(X_T^\pi\right)\},\ \text{ s.t. } X_k^\pi \text{ satisfies (1)},\tag{2}$$

where $\omega>0$ is the risk aversion level of the manager.

As said in the introduction, problem (2) is time-inconsistent and there are two main methods to handle it.

The first one is that fix an initial point $(k,X_k^\pi,\xi_k)=(0,x_0,i_0)$, and then try to find the optimal investment strategy $\hat{\pi}(0)$ for problem (2), simply disregarding whether the later parts of strategy $\hat{\pi}(0)$ are optimal or not. In previous literature, this strategy is called a pre-commitment strategy, which is a time-inconsistent but a global optimal strategy. In this case, the mean–variance model can be rewritten as

$$\begin{aligned}P(\omega):\quad&\max_{\pi(0)\in\Theta_0}\{\mathbb{E}_{0,x_0,i_0}\left(X_T^\pi\right)-\omega\operatorname{Var}_{0,x_0,i_0}\left(X_T^\pi\right)\},\\&\text{s.t. } X_k^\pi \text{ satisfies (1)}\end{aligned}\tag{3}$$

where $\mathbb{E}_{k,x_k,i}\left(X_T^\pi\right)=\mathbb{E}\left(X_T^\pi\mid X_k^\pi=x_k,\ \xi_k=i\right)$, $\operatorname{Var}_{k,x_k,i}\left(X_T^\pi\right)=\mathbb{E}_{k,x_k,i}\left(\left(X_T^\pi\right)^2\right)-\left(\mathbb{E}_{k,x_k,i}\left(X_T^\pi\right)\right)^2$, $X_0=x_0$, $\xi_0=i_0$.

The second one is that take the decision-making process as a non-cooperative game, and suppose that there is one decision maker at each time k $(k=0,1\ldots,T-1)$. At time k, under the current information (x_k,i), the decision maker can only choose the current control $\pi_k(i)$, and the controls in future time $k+1,\ldots,T-1$ are determined by the future decision makers. This decision-making process guarantees that the strategy starting from any time k is optimal, i.e., the strategy is time-consistent. But since the decision maker can only choose the current control, she cannot obtain the global optimal strategy. We call the corresponding strategy the (subgame perfect Nash) equilibrium strategy. In this case, the manager updates her target at each time k upon the information (x_k,i) at that time with the objective function

$$J_k\left(x_k,i,\pi(k)\right)=\mathbb{E}_{k,x_k,i}\left(X_T^\pi\right)-\omega\operatorname{Var}_{k,x_k,i}\left(X_T^\pi\right),\tag{4}$$

and solves a series of mean–variance models

$$\max_{\pi(k)\in\Theta_k} J_k\left(x_k, i, \pi(k)\right), \quad \text{s.t. } X_k^\pi \text{ satisfies (1).} \tag{5}$$

For convenience, for $k = 0, 1, \ldots, T-1$, any time-dependent $m \times 1$ vector H_l, we define $\sum_{l=t}^{k-1} H_l = \mathbf{0}$ for $t \geq k$, where $\mathbf{0}$ is the $m \times 1$ zero vector; any time-dependent $m \times m$ matrix N_l, we define $\prod_{l=k}^{k-1} N_l = \mathbf{I}$, where $\mathbf{I}$ is the $m \times m$ unit matrix. In particular, if $m = 1$, then $\sum_{l=t}^{k-1} H_l = 0$ and $\prod_{l=k}^{k-1} N_l = 1$.

3. Pre-commitment strategy and efficient frontier

In this section, we aim to derive the optimal strategy and efficient frontier of problem $P(\omega)$. Due to the non-separability of the variance operator, problem $P(\omega)$ cannot be directly dealt with by the dynamic programming method. Fortunately, by the embedding technique of Li and Ng (2000), problem $P(\omega)$ can be embedded into the following separable auxiliary problem:

$$A(\lambda, \omega): \quad \max_{\pi(0)\in\Theta_0} \left\{ \mathbb{E}_{0,x_0,i_0} \left(-\omega \left(X_T^\pi\right)^2 + \lambda X_T^\pi \right) \right\}, \quad \text{s.t. } X_k^\pi \text{ satisfies (1)} \tag{6}$$

where λ is an auxiliary parameter.

As showed in Li and Ng (2000), we can assert that the optimal strategy for problem $P(\omega)$ is among the optimal solutions for problems $A(\lambda, \omega)$ with different λ. In particular, an optimal solution of problem $P(\omega)$, if it exists, can be found by selecting $\lambda = 1 + 2\omega \mathbb{E}_{0,x_0,i_0}\left(X_T^{\hat{\pi}^A}\right)$, where $\mathbb{E}_{0,x_0,i_0}\left(X_T^{\hat{\pi}^A}\right)$ is the value corresponding to the optimal solution $\hat{\pi}^A$ of $A(\lambda, \omega)$. Therefore, obtaining the optimal strategy of problem $P(\omega)$ boils down to solving the problem $A(\lambda, \omega)$.

3.1. Solution of auxiliary problem $A(\lambda, \omega)$

By virtue of the separability of the auxiliary problem $A(\lambda, \omega)$, the dynamic programming method can be employed to obtain its optimal solution.

For $k = 0, 1, \ldots, T-1$ and $\xi_k = i \in \Pi$, define the value function

$$\begin{aligned} v_k(x_k, i) &= \max_{\pi(k)\in\Theta_k} \left\{ \mathbb{E}_{k,x_k,i} \left(-\omega \left(X_T^\pi\right)^2 + \lambda X_T^\pi \right) \right\} \\ &= \max_{\pi(k)\in\Theta_k} \left\{ \mathbb{E}\left(-\omega \left(X_T^\pi\right)^2 + \lambda X_T^\pi \,|X_k^\pi = x_k,\ \xi_k = i \right) \right\}. \end{aligned}$$

Then we have the Bellman equation

$$v_k(x_k, i) = \max_{\pi_k(i)} \left\{ \mathbb{E}\left(v_{k+1}\left(X_{k+1}^\pi, \xi_{k+1}\right) |X_k^\pi = x_k,\ \xi_k = i \right) \right\}. \tag{7}$$

According to the Markov state transition matrix, Eq. (7) can be rewritten as

$$\begin{aligned} v_k(x_k, i) &= \max_{\pi_k(i)} \left\{ \sum_{j=1}^J q_{ij}(k) \mathbb{E}\left(v_{k+1}\left(X_{k+1}^\pi, j\right) |X_k^\pi = x_k,\ \xi_k = i \right) \right\} \\ &= \max_{\pi_k(i)} \left\{ \sum_{j=1}^J q_{ij}(k) \mathbb{E}\left(v_{k+1}\left(A_{k,k} x_k + B_k(\beta) + \frac{S_k'(i)\pi_k(i)}{p_{k+y}}, j \right) \right) \right\} \end{aligned} \tag{8}$$

with terminal condition

$$v_T(x, i) = -\omega x^2 + \lambda x \text{ for all } i \in \Pi. \tag{9}$$

In order to solve the recursive equation (8), we introduce some notations and backward time series. For $k = 0, 1, \ldots, T-1$ and $i \in \Pi$, we define the following notations:

$$\Upsilon_k(i) = \mathbb{E}(S_k(i)S_k'(i)) = \mathbf{cov}_k(i) + s_k(i)s_k'(i), \tag{10}$$

$$h_k(i) = s_k'(i)\Upsilon_k^{-1}(i)s_k(i), \tag{11}$$

$$f_k(i) = 1 - h_k(i), \tag{12}$$

and construct the backward time series as follows:

$$M_k = \left(M_{k+1} - 2\omega A_{k+1,T-1}^2 B_k(\beta)\right) A_{k,k}, \tag{13}$$

$$\eta_k(i) = f_k(i) \sum_{j=1}^J q_{ij}(k)\eta_{k+1}(j), \tag{14}$$

$$D_k(i) = \frac{M_{k+1}^2}{4\omega A_{k+1,T-1}^2} h_k(i) \sum_{j=1}^J q_{ij}(k)\eta_{k+1}(j) + \sum_{j=1}^J q_{ij}(k) D_{k+1}(j) + \left(M_{k+1}B_k(\beta) - \omega A_{k+1,T-1}^2 B_k^2(\beta)\right)\eta_k(i) \tag{15}$$

with terminal condition

$$M_T = \lambda,\ \ \eta_T(i) = 1,\ \ D_T(i) = 0.$$

Remark 3.1. For $k = 0, 1, \ldots, T-1$ and $i \in \Pi$, since $\mathbf{cov}_k(i)$ is positive definite by Assumption 2.2, it is clear that $\Upsilon_k(i)$ is also positive definite. Furthermore, according to Lemma 2 of Çakmak and Özekici (2006), we have $0 < h_k(i) < 1$ and $0 < f_k(i) < 1$.

According to the recursive formula (13) and its terminal condition, we get the expression of M_k in the following lemma.

Lemma 3.2. *For all $k = 0, 1, \ldots, T$,*

$$M_k = \lambda A_{k,T-1} - 2\omega \sum_{l=k}^{T-1} B_l(\beta) A_{k,l} A_{l+1,T-1}^2. \tag{16}$$

Proof. See Appendix A. □

Next, we derive the expressions of $\eta_k(i)$ and $D_k(i)$.

For all $k = 0, 1, \ldots, T$, let

$$\boldsymbol{\eta}_\mathbf{k} = (\eta_k(1), \eta_k(2), \ldots, \eta_k(J))', \quad \mathbf{D_k} = (D_k(1), D_k(2), \ldots, D_k(J))',$$
$$\mathbf{f_k} = \mathbf{diag}(f_k(1), f_k(2), \ldots, f_k(J)), \quad \mathbf{h_k} = \mathbf{diag}(h_k(1), h_k(2), \ldots, h_k(J)),$$

where $\mathbf{diag}(a_1, a_2, \ldots, a_J)$ is a $J \times J$ diagonal matrix with elements $a_1, a_2, \ldots, a_J$. Then the recursive formulas (14) and (15) with their terminal conditions can be rewritten as

$$\boldsymbol{\eta}_\mathbf{k} = \mathbf{f_k} Q(k) \boldsymbol{\eta}_\mathbf{k+1},\ \ \boldsymbol{\eta}_\mathbf{T} = \mathbb{I}, \tag{17}$$

$$\mathbf{D_k} = \frac{M_{k+1}^2}{4\omega A_{k+1,T-1}^2} \mathbf{h_k} Q(k) \boldsymbol{\eta}_\mathbf{k+1} + Q(k)\mathbf{D_{k+1}} + \left(M_{k+1}B_k(\beta) - \omega A_{k+1,T-1}^2 B_k^2(\beta)\right)\boldsymbol{\eta}_\mathbf{k},\ \ \mathbf{D_T} = \mathbf{0}, \tag{18}$$

where $\mathbb{I} = (1, 1, \ldots, 1)'$ is a $J \times 1$ vector and $\mathbf{0}$ is the $J \times 1$ zero vector.

Lemma 3.3. *For all $k = 0, 1, \ldots, T$,*

$$\boldsymbol{\eta}_\mathbf{k} = \left(\prod_{m=k}^{T-1} \mathbf{f_m} Q(m) \right) \mathbb{I}, \tag{19}$$

$$\mathbf{D_k} = \sum_{m=k}^{T-1} \frac{M_{m+1}^2}{4\omega A_{m+1,T-1}^2} \left(\prod_{l=k}^{m-1} Q(l) \right) \mathbf{h_m} Q(m) \boldsymbol{\eta}_\mathbf{m+1} + \sum_{m=k}^{T-1} \left(M_{m+1}B_m(\beta) - \omega A_{m+1,T-1}^2 B_m^2(\beta)\right) \left(\prod_{l=k}^{m-1} Q(l) \right) \boldsymbol{\eta}_\mathbf{m}. \tag{20}$$

Proof. See Appendix B. □

After getting the expressions of $\boldsymbol{\eta}_{\mathbf{k}}$ and $\mathbf{D}_{\mathbf{k}}$, their ith elements give the expressions of $\eta_k(i)$ and $D_k(i)$.

Lemma 3.4. *For all $k = 0, 1, \ldots, T-1$ and all $i \in \Pi$,*

$$0 < \eta_k(i) < 1. \tag{21}$$

Proof. See Appendix C. □

Based on the above preliminary results, we can now solve the auxiliary problem $A(\lambda, \omega)$.

Theorem 3.5. *For $k = 0, 1, \ldots, T-1$, the optimal value function of auxiliary problem $A(\lambda, \omega)$ is*

$$v_k(x_k, i) = -\omega A^2_{k,T-1}\eta_k(i)x_k^2 + M_k\eta_k(i)x_k + D_k(i), \tag{22}$$

and the corresponding optimal strategy is

$$\begin{aligned}\hat{\pi}_k^A(i) &= \left(\frac{M_{k+1}}{2\omega A^2_{k+1,T-1}} - A_{k,k}x_k - B_k(\beta)\right) p_{k+y}\Upsilon_k^{-1}(i)s_k(i)\\ &= \left(-\sum_{l=k}^{T-1}\frac{B_l(\beta)}{A_{k+1,l}} + \frac{\lambda}{2\omega A_{k+1,T-1}} - A_{k,k}x_k\right) p_{k+y}\Upsilon_k^{-1}(i)s_k(i), \end{aligned}\tag{23}$$

where M_k, $\eta_k(i)$ and $D_k(i)$ are given in Lemmas 3.2 and 3.3.

Proof. See Appendix D. □

3.2. Solution and efficient frontier of original problem $P(\omega)$

Before moving on to solve the problem $P(\omega)$ for its solution and efficient frontier, we define some notations first. For $k, t = 0, 1, \ldots, T-1$, define

$$\chi_k(\beta) = B_k(\beta)A_{k+1,T-1}, \tag{24}$$

$$\zeta(\beta) = A_{0,T-1}\sum_{l=0}^{T-1}\chi_l(\beta), \tag{25}$$

$$a_t(i_t) = \sum_{l=t}^{T-1}\phi_l(i_t), \tag{26}$$

$$\theta_k(i_t) = \mathbb{E}\left(\prod_{l=k}^{T-1} f_l(\xi_l)|\xi_t = i_t,\ t \le k\right), \tag{27}$$

$$\phi_k(i_t) = \mathbb{E}\left(h_k(\xi_k)\prod_{l=k+1}^{T-1} f_l(\xi_l)|\xi_t = i_t,\ t \le k\right), \tag{28}$$

$$b_k(\beta, i_0) = \sum_{l=0}^{k}\theta_l(i_0)\chi_l(\beta) - \sum_{l=0}^{k}\phi_l(i_0)\sum_{m=l+1}^{T-1}\chi_m(\beta), \tag{29}$$

$$\begin{aligned}\psi(\beta, i_0) &= \sum_{l=0}^{T-1}\chi_l^2(\beta)\theta_l(i_0) + \sum_{l=0}^{T-1}\left(\sum_{m=l+1}^{T-1}\chi_m(\beta)\right)^2\phi_l(i_0)\\ &\quad + 2\sum_{l=0}^{T-1}\chi_l(\beta)b_{l-1}(\beta, i_0).\end{aligned}\tag{30}$$

The following lemma gives the expressions of $\theta_k(i_t)$ and $\phi_k(i_t)$.

Lemma 3.6. *For $t, k = 0, 1, \ldots, T-1$, $t \le k$ and $\xi_t = i_t \in \Pi$, we have*

$$\theta_k(i_t) = \left(\left(\prod_{l=t}^{k-1}Q(l)\right)\boldsymbol{\eta}_{\mathbf{k}}\right)(i_t), \tag{31}$$

$$\phi_k(i_t) = \left(\left(\prod_{l=t}^{k-1}Q(l)\right)\mathbf{h}_{\mathbf{k}}Q(k)\boldsymbol{\eta}_{\mathbf{k+1}}\right)(i_t). \tag{32}$$

Proof. See Appendix E. □

Remark 3.7. From Lemma 3.6, we have $\theta_0(i_0) = \eta_0(i_0)$. In view of Lemma 2.1 in Chen et al. (2016), we know $a_0(i_0) = 1 - \theta_0(i_0)$. It is an immediate consequence of Lemma 3.4 that $0 < a_0(i_0) < 1$.

By using the similar method as that in Lemma 2.1 of Chen et al. (2016), we can easily obtain the following lemma.

Lemma 3.8. *For $k = 0, 1, \ldots, T-1$ and $\xi_0 = i_0 \in \Pi$, we have $\phi_k(i_0) = \theta_{k+1}(i_0) - \theta_k(i_0)$.*

The following lemma holds as a result of Lemma 3.8.

Lemma 3.9. *For $k = 0, 1, \ldots, T-1$ and $\xi_0 = i_0 \in \Pi$, we have*

$$b_k(\beta, i_0) = \theta_0(i_0)\sum_{m=0}^{T-1}\chi_m(\beta) - \theta_{k+1}(i_0)\sum_{m=k+1}^{T-1}\chi_m(\beta).$$

In particular, $b_{T-1}(\beta, i_0) = \theta_0(i_0)\sum_{m=0}^{T-1}\chi_m(\beta)$.

Proof. See Appendix F. □

In order to obtain the optimal solution and the efficient frontier of problem $P(\omega)$, we need to calculate $\mathbb{E}_{0,x_0,i_0}\left(X_T^{\hat{\pi}^A}\right)$ and $\mathbb{E}_{0,x_0,i_0}\left(\left(X_T^{\hat{\pi}^A}\right)^2\right)$.

Theorem 3.10. *For given initial state $\xi_0 = i_0$ and initial wealth $X_0 = x_0$, we have*

$$\mathbb{E}_{0,x_0,i_0}\left(X_T^{\hat{\pi}^A}\right) = A_{0,T-1}\theta_0(i_0)x_0 + b_{T-1}(\beta, i_0) + \frac{\lambda}{2\omega}a_0(i_0), \tag{33}$$

$$\begin{aligned}\mathbb{E}_{0,x_0,i_0}\left(\left(X_T^{\hat{\pi}^A}\right)^2\right) &= A^2_{0,T-1}\theta_0(i_0)x_0^2 + 2\zeta(\beta)\theta_0(i_0)x_0\\ &\quad + \frac{\lambda^2}{4\omega^2}a_0(i_0) + \psi(\beta, i_0).\end{aligned}\tag{34}$$

Proof. See Appendix G. □

Now, following Theorems 3.5 and 3.10, the solution and efficient frontier of problem $P(\omega)$ can be derived.

Theorem 3.11. *Suppose the initial state $\xi_0 = i_0$ and the initial wealth $X_0 = x_0$. For $k = 0, 1, \ldots, T-1$, let $X_k^{\pi} = x_k$ and $\xi_k = i \in \Pi$. The optimal strategy of problem $P(\omega)$ is given by*

$$\begin{aligned}\hat{\pi}_k^P(i) &= \left(\sum_{l=0}^{k-1}B_l(\beta)A_{l+1,k} + \frac{1}{2\omega\eta_0(i_0)A_{k+1,T-1}} + A_{0,k}x_0 - A_{k,k}x_k\right)\\ &\quad \times p_{k+y}\Upsilon_k^{-1}(i)s_k(i)\\ &= \left(\sum_{l=0}^{k-1}\frac{\left(C_lr_l - \beta q_{l+y}\sum_{m=0}^{l}C_m\right)\prod_{m=l+1}^{k}r_m}{\prod_{m=l}^{k-1}p_{m+y}} + \frac{\prod_{l=k}^{T-1}p_{l+y}}{2\omega\eta_0(i_0)\prod_{l=k+1}^{T-1}r_l}\right.\\ &\quad \left. + \frac{\prod_{l=0}^{k}r_l}{\prod_{l=0}^{k-1}p_{l+y}}x_0 - r_kx_k\right)\Upsilon_k^{-1}(i)s_k(i),\end{aligned}\tag{35}$$

and the efficient frontier is

$$\begin{aligned}\mathrm{Var}_{0,x_0,i_0}\left(X_T^{\hat{\pi}^P}\right) &= \frac{\eta_0(i_0)}{a_0(i_0)}\left[\mathbb{E}_{0,x_0,i_0}\left(X_T^{\hat{\pi}^P}\right) - A_{0,T-1}x_0 - \sum_{l=0}^{T-1}\chi_l(\beta)\right]^2\\ &\quad - \eta_0(i_0)\left(\sum_{l=0}^{T-1}\chi_l(\beta)\right)^2 + \psi(\beta, i_0).\end{aligned}\tag{36}$$

Proof. See Appendix H. □

Remark 3.12. Eq. (35) indicates that: (i) at any time, the pension manager will invest less wealth in the risky assets as her risk aversion level becomes larger; (ii) at any time k, the portfolio $\hat{\pi}_k^P(i)$ is proportional to the vector $\Upsilon_k^{-1}(i)s_k(i)$, which implies that the well-known two funds separation theorem holds; (iii) the investment $\hat{\pi}_k^P(i)$ at any time k depends not only on the current wealth x_k and state i, but also on the initial wealth x_0 and state i_0; (iv) the current portfolio $\hat{\pi}_k^P(i)$ is affected by all the survival rates p_{l+y} and interest rates r_l, $l = 0, 1, \ldots, T-1$ and by the contributions C_m, $m = 0, 1, \ldots, k-1$ before the current time; (v) the return of premiums clause has a significant effect on the pre-commitment strategy and the effect is increasing with time (this is because the mortality rate and the contribution accumulation are increasing with time, and hence so are the amount of the premiums returned to the dead members).

4. Equilibrium strategy and efficient frontier

In this section, we will solve problem (5) to obtain an equilibrium strategy.

Definition 4.1 (*Equilibrium Strategy*). Let $\hat{\pi}^E$ be a given time-0 admissible strategy. For an arbitrary point (k, x_k, i) and an arbitrary decision $\pi_k(i)$ adapted to $\wp_k$, define the time-k admissible strategy

$$\widetilde{\pi}(k) = \left(\pi_k(i), \hat{\pi}_{k+1}^E(\xi_{k+1}), \ldots, \hat{\pi}_{T-1}^E(\xi_{T-1})\right).$$

Then $\hat{\pi}^E$ is said to be a subgame perfect Nash equilibrium strategy (equilibrium strategy for short) if for every k, it satisfies

$$\max_{\pi_k(i)} J_k\left(x_k, i, \tilde{\pi}(k)\right) = J_k\left(x_k, i, \hat{\pi}^E(k)\right),$$

where $\hat{\pi}^E(k) = \left(\hat{\pi}_k^E(i), \hat{\pi}_{k+1}^E(\xi_{k+1}), \ldots, \hat{\pi}_{T-1}^E(\xi_{T-1})\right)$. Furthermore, if an equilibrium strategy $\hat{\pi}^E$ exists, the equilibrium value function is defined as

$$V_k\left(x_k, i\right) = J_k\left(x_k, i, \hat{\pi}^E(k)\right).$$

From Definition 4.1, finding an equilibrium strategy, at any time k, for any given $X_k^\pi = x_k$ and state $\xi_k = i$, amounts to solving the following problem:

$$\begin{aligned} V_k\left(x_k, i\right) =& J_k\left(x_k, i, \hat{\pi}^E(k)\right) \\ =& \max_{\pi_k(i)} J_k\left(x_k, i, \left(\pi_k(i), \hat{\pi}_{k+1}^E\left(\xi_{k+1}\right), \ldots, \hat{\pi}_{T-1}^E\left(\xi_{T-1}\right)\right)\right). \end{aligned} \tag{37}$$

4.1. Equilibrium strategy

In order to get an equilibrium strategy and an equilibrium value function, we apply a backward induction method to the equilibrium value function $V_k(x_k, i)$.

Fix an arbitrarily chosen initial point (k, x_k, i) and denote

$$g_k(x_k, i) = \mathbb{E}_{k,x_k,i}\left[X_T^{\hat{\pi}^E}\right]. \tag{38}$$

Then, from Björk and Murgoci (2010, 2014), the equilibrium value function satisfies the extended Bellman equation

$$\begin{aligned} V_k(x_k, i) =& \max_{\pi_k(i)} \Big\{\mathbb{E}_{k,x_k,i}\left(V_{k+1}\left(X_{k+1}^\pi, \xi_{k+1}\right)\right) \\ &- \omega\mathbb{E}_{k,x_k,i}\left(g_{k+1}^2\left(X_{k+1}^\pi, \xi_{k+1}\right)\right) \\ &+ \omega\Big[\mathbb{E}_{k,x_k,i}\left(g_{k+1}\left(X_{k+1}^\pi, \xi_{k+1}\right)\right)\Big]^2\Big\}, \quad V_T(x, i) = x, \end{aligned} \tag{39}$$

where

$$g_k(x_k, i) = \mathbb{E}_{k,x_k,i}\left[g_{k+1}\left(X_{k+1}^\pi, \xi_{k+1}\right)\right], \quad g_T(x, i) = x. \tag{40}$$

To derive an expression of the equilibrium value function, for $k = 0, 1, \ldots, T-1$ and $i \in \Pi$, we define a notation

$$z_k(i) = s_k'(i)\mathbf{cov}_k^{-1}(i)s_k(i), \tag{41}$$

and construct two backward time series

$$\varpi_k(i) = z_k(i) + \sum_{j=1}^J q_{ij}(k)\varpi_{k+1}(j), \tag{42}$$

$$\begin{aligned} W_k(i) =& \sum_{j=1}^J q_{ij}(k)W_{k+1}(j) + \sum_{j=1}^J q_{ij}(k)\varpi_{k+1}^2(j) \\ &- \left(\sum_{j=1}^J q_{ij}(k)\varpi_{k+1}(j)\right)^2, \end{aligned} \tag{43}$$

with terminal condition

$$\varpi_T(i) = 0, \quad W_T(i) = 0.$$

We first derive the expressions of $\varpi_k(i)$ and $W_k(i)$. For $k = 0, 1, \ldots, T-1$, let

$$\boldsymbol{\varpi}_\mathbf{k} = (\varpi_k(1), \varpi_k(2), \ldots, \varpi_k(J))', \quad \mathbf{W_k} = (W_k(1), W_k(2), \ldots, W_k(J))',$$

$$\mathbf{z_k} = (z_k(1), z_k(2), \ldots, z_k(J))',$$
$$\boldsymbol{\varpi}_\mathbf{k}^2 = \left((\varpi_k(1))^2, \ (\varpi_k(2))^2, \ldots, (\varpi_k(J))^2\right)',$$

$$\begin{aligned} &(Q(k)\boldsymbol{\varpi}_\mathbf{k+1})^2 \\ &= \left(((Q(k)\boldsymbol{\varpi}_\mathbf{k+1})(1))^2, ((Q(k)\boldsymbol{\varpi}_\mathbf{k+1})(2))^2, \ldots, ((Q(k)\boldsymbol{\varpi}_\mathbf{k+1})(J))^2\right)'. \end{aligned}$$

Then the recursive formulas (42) and (43) with their terminal condition can be rewritten as

$$\boldsymbol{\varpi}_\mathbf{k} = \mathbf{z_k} + Q(k)\boldsymbol{\varpi}_\mathbf{k+1}, \quad \boldsymbol{\varpi}_\mathbf{T} = \mathbf{0}, \tag{44}$$

$$\mathbf{W_k} = Q(k)\mathbf{W_{k+1}} + Q(k)\boldsymbol{\varpi}_\mathbf{k+1}^2 - (Q(k)\boldsymbol{\varpi}_\mathbf{k+1})^2, \quad \mathbf{W_T} = \mathbf{0}. \tag{45}$$

Lemma 4.2. *For all* $k = 0, 1, \ldots, T$,

$$\boldsymbol{\varpi}_\mathbf{k} = \sum_{m=k}^{T-1}\left(\prod_{j=k}^{m-1} Q(j)\right)\mathbf{z_m}, \tag{46}$$

$$\begin{aligned} \mathbf{W_k} =& \sum_{m=k+1}^{T-1}\left(\prod_{j=k}^{m-1} Q(j)\right)\boldsymbol{\varpi}_\mathbf{m}^2 \\ &- \sum_{m=k+1}^{T-1}\left(\prod_{j=k}^{m-2} Q(j)\right)(Q(m-1)\boldsymbol{\varpi}_\mathbf{m})^2. \end{aligned} \tag{47}$$

Proof. See Appendix I. □

After getting the expressions of $\boldsymbol{\varpi}_\mathbf{k}$ and $\mathbf{W_k}$, their ith components give the expressions of $\varpi_k(i)$ and $W_k(i)$.

Remark 4.3. For all $k = 0, 1, \ldots, T-1$ and all $i \in \Pi$, since $\mathbf{cov}_k(i)$ is positive definite and $s_k(i) \neq \vec{0}_n$ by Assumptions 2.2 and 2.3, we have $z_k(i) > 0$, and hence $\varpi_k(i) > 0$ by Lemma 4.2.

Now we can give the equilibrium strategy and the equilibrium value function.

Theorem 4.4. *For* $k = 0, 1, \ldots, T-1$, $X_k^\pi = x_k$ *and* $\xi_k = i \in \Pi$, *the optimal equilibrium strategy is given by*

$$\hat{\pi}_k^E(i) = \frac{p_{k+y}}{2\omega A_{k+1,T-1}}\mathbf{cov}_k^{-1}(i)s_k(i) = \frac{\prod_{l=k}^{T-1} p_{l+y}}{2\omega\prod_{l=k+1}^{T-1} r_l}\mathbf{cov}_k^{-1}(i)s_k(i), \tag{48}$$

the equilibrium value function is given by

$$V_k(x_k, i) = A_{k,T-1}x_k + \sum_{l=k}^{T-1} \chi_l(\beta) + \frac{1}{4\omega}(\varpi_k(i) - W_k(i)), \tag{49}$$

and

$$g_k(x_k, i) = A_{k,T-1}x_k + \sum_{l=k}^{T-1} \chi_l(\beta) + \frac{1}{2\omega}\varpi_k(i). \tag{50}$$

Proof. See Appendix J. □

Remark 4.5. From Theorem 4.4 we find that: (i) in contrast to the pre-commitment strategy, the equilibrium strategy is independent of the wealth, the contribution and even the return of premiums clause at any time. This result is unrealistic from an economic point of view as pointed out in Björk et al. (2014). That is, in order to achieve time-consistency, some important factors are ignored in the equilibrium strategy. From this perspective, the pre-commitment strategy is more practical than the equilibrium strategy; (ii) the portfolio $\hat{\pi}_k^E(i)$ at any time depends on the current state, the future interest rates and the future survival rates, but is independent of the initial state, the past interest rates and the past survival rates. This is quite different from the pre-commitment strategy. The reason is that, the equilibrium investor aims to find the time-consistent strategy at any time k based on the forthcoming information while the pre-commitment investor aims to find the globally optimal strategy from the viewpoint of initial time; (iii) $V_k|_{\beta=1} < V_k|_{\beta=0}$ for all $k = 0, 1, \ldots, T-1$, that is, the equilibrium value function with the return of premiums clause is less than the one without the return of premiums clause. The reason is that when the return of premiums clause is considered, a part of the wealth will be returned to the members who die before retirement, and thus the wealth is less than the case without the return of premiums clause.

4.2. Equilibrium efficient frontier

We consider the efficient frontier starting from arbitrary initial point (k, x_k, i) with time $k \in \{0, 1, \ldots, T-1\}$, wealth $X_k^{\pi} = x_k$ and state $\xi_k = i \in \Pi$. By Eqs. (38), (49) and (50), we have

$$\mathbb{E}_{k,x_k,i}\left(X_T^{\hat{\pi}^E}\right) = g_k(x_k, i) = A_{k,T-1}x_k + \sum_{l=k}^{T-1} \chi_l(\beta) + \frac{1}{2\omega}\varpi_k(i), \tag{51}$$

and

$$\begin{aligned} V_k(x_k, i) &= J_k\left(x_k, i, \hat{\pi}^E(k)\right) = \mathbb{E}_{k,x_k,i}\left(X_T^{\hat{\pi}^E}\right) - \omega \mathrm{Var}_{k,x_k,i}\left(X_T^{\hat{\pi}^E}\right) \\ &= A_{k,T-1}x_k + \sum_{l=k}^{T-1} \chi_l(\beta) + \frac{1}{2\omega}\varpi_k(i) - \omega \mathrm{Var}_{k,x_k,i}\left(X_T^{\hat{\pi}^E}\right) \\ &= A_{k,T-1}x_k + \sum_{l=k}^{T-1} \chi_l(\beta) + \frac{1}{4\omega}(\varpi_k(i) - W_k(i)). \end{aligned} \tag{52}$$

Hence,

$$\mathrm{Var}_{k,x_k,i}\left(X_T^{\hat{\pi}^E}\right) = \frac{1}{4\omega^2}(\varpi_k(i) + W_k(i)). \tag{53}$$

Since $\varpi_k(i) > 0$ from Remark 4.3, Eq. (51) yields

$$\frac{1}{2\omega} = \frac{\mathbb{E}_{k,x_k,i}\left(X_T^{\hat{\pi}^E}\right) - A_{k,T-1}x_k - \sum_{l=k}^{T-1} \chi_l(\beta)}{\varpi_k(i)}. \tag{54}$$

Substituting Eq. (54) into Eq. (53), we obtain the efficient frontier

$$\begin{aligned} &\mathrm{Var}_{k,x_k,i}\left(X_T^{\hat{\pi}^E}\right) \\ &= \frac{\left(\mathbb{E}_{k,x_k,i}\left(X_T^{\hat{\pi}^E}\right) - A_{k,T-1}x_k - \sum_{l=k}^{T-1} \chi_l(\beta)\right)^2}{(\varpi_k(i))^2}(\varpi_k(i) + W_k(i)). \end{aligned} \tag{55}$$

In the equilibrium efficient frontier, the global minimum variance equals zero. Moreover, the equilibrium efficient frontier without the return of premiums clause lies above the one with the clause, that is to say, to obtain the same expected terminal wealth, the pension manager with the return of premiums clause needs to bear more risk than the case without the clause. It can be interpreted this way: when there is a return of premiums clause, the manager needs to return part of the accumulated wealth to the members who die during the accumulation period, which increases her risk.

5. Special cases

Special case 1: There is no regime switching. In this case, there is only one state and $Q(k) = 1$ for $k = 0, 1, \ldots, T-1$. Then,

$$\theta_k = \eta_k = \prod_{l=k}^{T-1}(1 - s_l'\Upsilon_l^{-1}s_l), \tag{56}$$

$$a_0 = 1 - \prod_{l=0}^{T-1}(1 - s_l'\Upsilon_l^{-1}s_l), \tag{57}$$

$$\phi_k = s_k'\Upsilon_k^{-1}s_k \prod_{l=k+1}^{T-1}(1 - s_l'\Upsilon_l^{-1}s_l), \tag{58}$$

$$b_k(\beta) = \theta_0 \sum_{l=0}^{T-1} \chi_l(\beta) - \theta_{k+1} \sum_{l=k+1}^{T-1} \chi_l(\beta), \tag{59}$$

$$\begin{aligned} \psi(\beta) &= \sum_{l=0}^{T-1} \chi_l^2(\beta)\theta_l + \sum_{l=0}^{T-1}\left(\sum_{m=l+1}^{T-1} \chi_m(\beta)\right)^2 \phi_l \\ &\quad + 2\sum_{l=0}^{T-1} \chi_l(\beta) b_{l-1}(\beta), \end{aligned} \tag{60}$$

$$\varpi_k = \sum_{l=k}^{T-1}\left(s_l'\mathbf{cov}_l^{-1}s_l\right), \tag{61}$$

$$W_k = 0. \tag{62}$$

Hence, the pre-commitment strategy and the corresponding efficient frontier can be simplified as

$$\begin{aligned} \hat{\pi}_k^P = &\left(\sum_{l=0}^{k-1} B_l(\beta)A_{l+1,k} + \frac{1}{2\omega\prod_{l=0}^{T-1}(1 - s_l'\Upsilon_l^{-1}s_l)A_{k+1,T-1}}\right. \\ &\left. + A_{0,k}x_0 - A_{k,k}x_k\right) p_{k+y}\Upsilon_k^{-1}s_k, \end{aligned} \tag{63}$$

$$\begin{aligned} \mathrm{Var}_{0,x_0}\left(X_T^{\hat{\pi}^P}\right) = &\frac{\eta_0}{a_0}\left[\mathbb{E}_{0,x_0}\left(X_T^{\hat{\pi}^P}\right) - A_{0,T-1}x_0 - \sum_{l=0}^{T-1} \chi_l(\beta)\right]^2 \\ &- \eta_0\left(\sum_{l=0}^{T-1} \chi_l(\beta)\right)^2 + \psi(\beta). \end{aligned} \tag{64}$$

The equilibrium strategy and the corresponding efficient frontier become as

$$\hat{\pi}_k^E = \frac{\prod_{l=k}^{T-1} p_{l+y}}{2\omega\prod_{l=k+1}^{T-1} r_l}\mathbf{cov}_k^{-1}s_k, \tag{65}$$

$$\mathrm{Var}_{k,x_k}\left(X_T^{\hat{\pi}^E}\right) = \frac{\left(\mathbb{E}_{k,x_k}\left(X_T^{\hat{\pi}^E}\right) - A_{k,T-1}x_k - \sum_{l=k}^{T-1}\chi_l(\beta)\right)^2}{\sum_{l=k}^{T-1}\left(s_l'\mathbf{cov}_l^{-1}s_l\right)}. \quad (66)$$

Special case 2: There is no return of premiums clause, that is to say, the pension plan members will be empty-handed if they die during the accumulation phase. In this case, $\beta = 0$. Then, for $k = 0, 1, \ldots, T-1$,

$$B_k(\beta) = B_k = C_k A_{k,k}, \quad (67)$$

$$\chi_k(\beta) = \chi_k = C_k A_{k,T-1}, \quad (68)$$

$$\zeta(\beta) = \zeta = A_{0,T-1}\sum_{l=0}^{T-1} C_l A_{l,T-1}, \quad (69)$$

$$b_k(\beta, i_0) = b_k(i_0) = \theta_0(i_0)\sum_{l=0}^{T-1} C_l A_{l,T-1} - \theta_{k+1}(i_0)\sum_{l=k+1}^{T-1} C_l A_{l,T-1}, \quad (70)$$

$$\psi(\beta, i_0) = \psi(i_0) = \sum_{l=0}^{T-1} C_l^2 A_{l,T-1}^2 \theta_l(i_0)$$

$$+ \sum_{l=0}^{T-1}\left(\sum_{m=l+1}^{T-1} C_m A_{m,T-1}\right)^2 \phi_l(i_0) + 2\sum_{l=0}^{T-1} C_l A_{l,T-1} b_{l-1}(i_0). \quad (71)$$

Hence, the pre-commitment strategy and the corresponding efficient frontier can be simplified as

$$\hat{\pi}_k^P = \left(\sum_{l=0}^{k-1}\frac{C_l\prod_{m=l}^{k} r_m}{\prod_{m=l}^{k-1} p_{m+y}} + \frac{\prod_{l=k}^{T-1} p_{l+y}}{2\omega\eta_0(i_0)\prod_{l=k+1}^{T-1} r_l} + \frac{\prod_{l=0}^{k} r_l}{\prod_{l=0}^{k-1} p_{l+y}}x_0 - r_k x_k\right)\Upsilon_k^{-1}(i)s_k(i), \quad (72)$$

$$\mathrm{Var}_{0,x_0,i_0}\left(X_T^{\hat{\pi}^P}\right) = \frac{\eta_0(i_0)}{a_0(i_0)}\left[\mathbb{E}_{0,x_0,i_0}\left(X_T^{\hat{\pi}^P}\right) - A_{0,T-1}x_0 - \sum_{l=0}^{T-1} C_l A_{l,T-1}\right]^2 - \eta_0(i_0)\left(\sum_{l=0}^{T-1} C_l A_{l,T-1}\right)^2 + \psi(i_0). \quad (73)$$

The equilibrium strategy in this case is also Eq. (48), and the corresponding efficient frontier changes to be

$$\mathrm{Var}_{k,x_k,i}\left(X_T^{\hat{\pi}^E}\right) = \frac{\left(\mathbb{E}_{k,x_k,i}\left(X_T^{\hat{\pi}^E}\right) - A_{k,T-1}x_k - \sum_{l=k}^{T-1} C_l A_{l,T-1}\right)^2}{(\varpi_k(i))^2}\left(\varpi_k(i) + W_k(i)\right). \quad (74)$$

6. Numerical example

In this section, a numerical example is presented to illustrate our results. The data used here is from the American market.

Consider a DC pension plan in which the accumulation process of a member starts from age 50 and ends at age 60, i.e., $y = 50$ and $T = 10$. Assume that the initial amount of her fund account is $x_0 = 1$ and she contributes $C_k = 1$ $(k = 0, 1, \ldots, 9)$ as a premium at the beginning of every year. The pension fund is managed by a manager with risk aversion level $\omega = 2$.

Suppose that the pension fund can be invested in a risk-free asset and three risky assets in the American market. These three risky assets (stocks) are COCA COLA CO (11 308), GENERAL ELECTRIC CO (12 060), INTERNATIONAL BUSINESS MACHS COR (12 490) (labeled by stocks 1, 2, 3). Our data set is composed of the historical annual returns of the three stocks from 1931 to 2016, with a sample size 86. Next, we choose average interest rates on American 5-year treasury bonds as the risk-free interest. Following Chen and Yang (2011) and Yao et al. (2016a, b), we roughly divide the market states into two regimes: the bearish ($i = 1$) and the bullish ($i = 2$). The states of the Markov chain are classified according to the average annual return of the three stocks. If the average annual return is less than the empirical median (based on the above historical data) of the average return, the state of the Markov chain is said to be in State 1, otherwise, it is said to be in State 2. Based on the data set above, for $k = 0, 1, \ldots, 9$ and $i = 1, 2$, we obtain the related parameters as follows

$$r_k = 1.0264, \quad s_k(1) = (-0.1136, \ -0.0712, \ -0.0824)',$$

$$s_k(2) = (0.1005, \ 0.0849, \ 0.1333)',$$

$$\mathbf{cov}_k(1) = \begin{bmatrix} 0.0612 & -0.0101 & 0.0043 \\ -0.0101 & 0.0802 & -0.0128 \\ 0.0043 & -0.0128 & 0.0792 \end{bmatrix},$$

$$\mathbf{cov}_k(2) = \begin{bmatrix} 0.0640 & -0.0017 & -0.0083 \\ -0.0017 & 0.0474 & 0.0060 \\ -0.0083 & 0.0060 & 0.0712 \end{bmatrix}.$$

Using the above historical data, we now derive the state transition probability matrix Q of the Markov chain. According to the classification of the market states, there are 43 years in State 1. Among all these 43 years being in State 1, we find that the number of the next year in State 1 is 17, and the number of the next year in State 2 is 26. Therefore, we calculate the empirical state transition probabilities $q_{11}(k)$ and $q_{12}(k)$ as follows

$$q_{11}(k) = 17/43 \approx 0.3953, \quad q_{12}(k) = 26/43 \approx 0.6047.$$

Similarly, we can calculate other empirical state transition probabilities $q_{21}(k) = 25/43 \approx 0.5814$ and $q_{22}(k) = 18/43 \approx 0.4186$. Hence the state transition probability matrix is

$$Q_k = \begin{bmatrix} q_{11}(k) & q_{12}(k) \\ q_{21}(k) & q_{22}(k) \end{bmatrix} = \begin{bmatrix} 0.3953 & 0.6047 \\ 0.5814 & 0.4186 \end{bmatrix}.$$

When the regime switching is not considered, we assume that the market has only one state. We use all the data listed above to estimate the parameters, and obtain

$$s_k = (-0.0066, \ 0.0069, \ 0.0255)',$$

$$\mathbf{cov}_k = \begin{bmatrix} 0.0885 & 0.0230 & 0.0296 \\ 0.0230 & 0.0711 & 0.0228 \\ 0.0296 & 0.0228 & 0.0931 \end{bmatrix}.$$

In addition, q_{k+y}, $k = 0, 1, \ldots, 10$ are from the life table of USA in 2015 as shown in Table 1.

Table 1
Life table of USA in 2015. [a]

k	0	1	2	3	4	5
q_{k+y}	0.00408	0.00448	0.00490	0.00534	0.00582	0.00621
k	6	7	8	9	10	
q_{k+y}	0.00676	0.00737	0.00788	0.00837	0.00893	

[a] Date from: http://www.mortality.org/.

6.1. Numerical analysis for the two investment strategies

This subsection analyzes the influence of regime switching and the return of premiums clause on the two investment strategies. For convenience, suppose that the market states evolve following Table 2.

Fig. 1 plots the two investment strategies. We find that no matter for the pre-commitment or the equilibrium strategy, the amounts invested in the risky assets in bull markets are more

Table 2
Market states from time 0 to $T-1$.

k	0	1	2	3	4	5	6	7	8	9
Market state	2	1	1	2	2	1	2	1	2	1

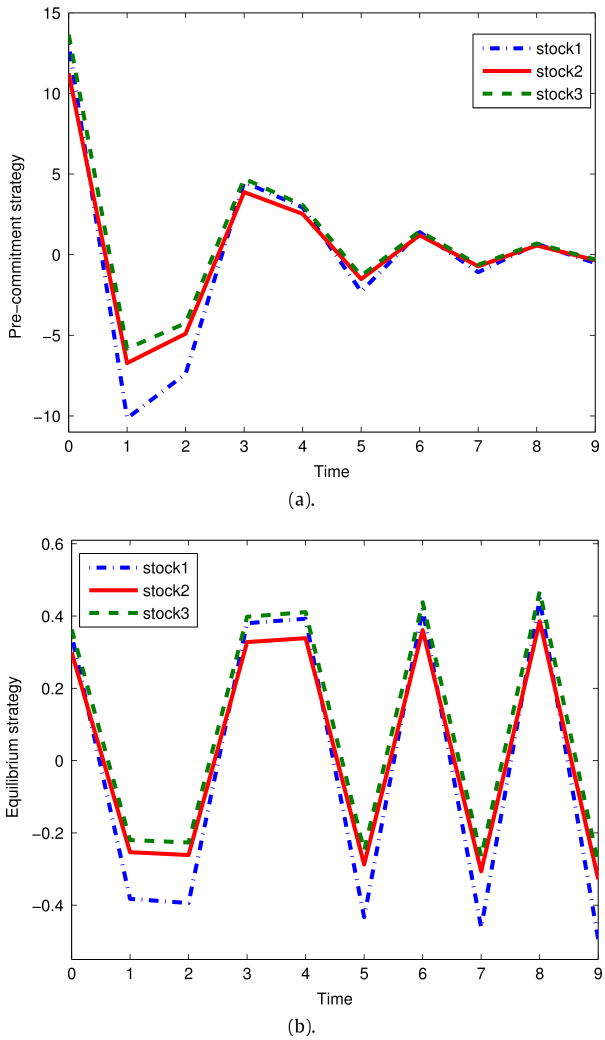


Fig. 1. The process of pre-commitment strategy and equilibrium strategy.

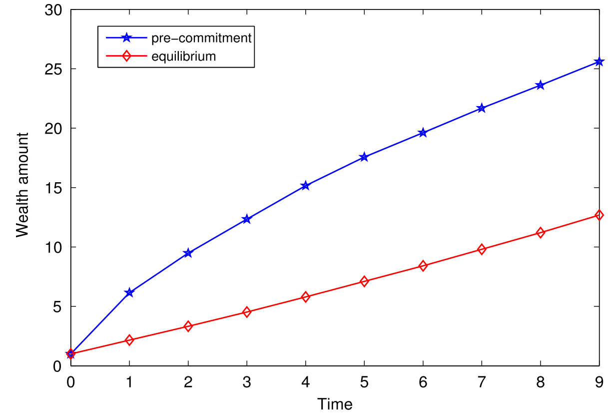


Fig. 2. The wealth process of two strategies.

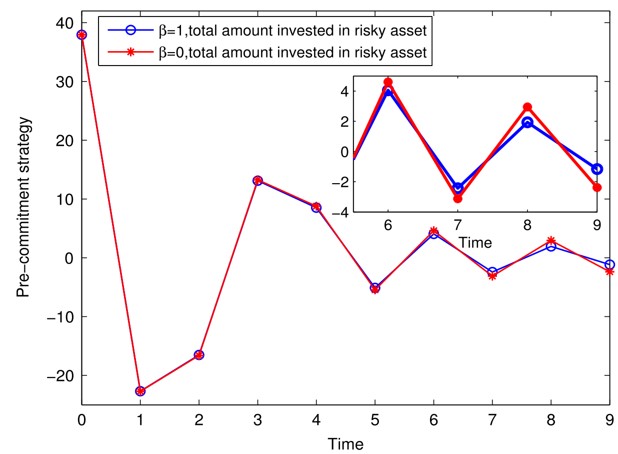
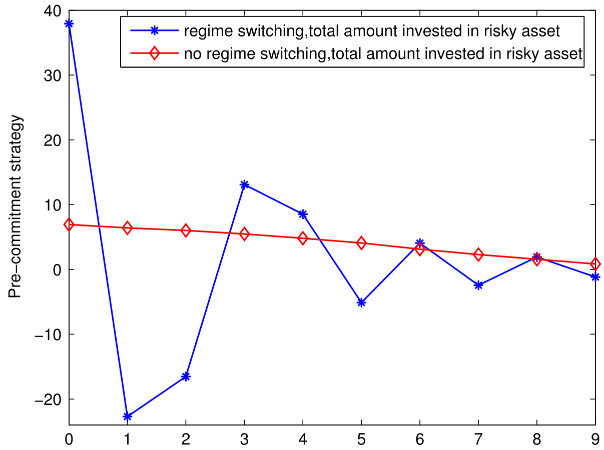


Fig. 3. The effect of the return of premiums clause on the pre-commitment strategy.

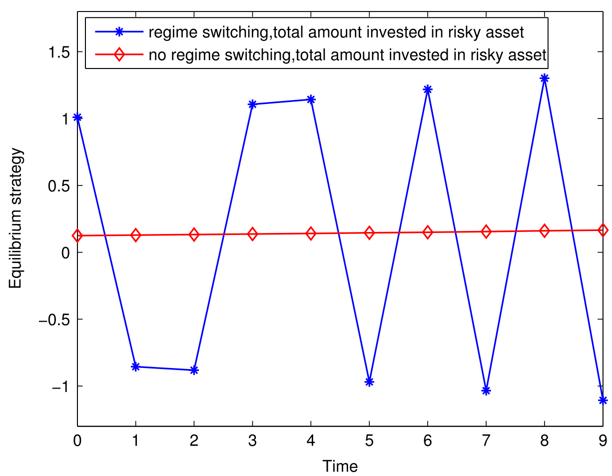
than those in bear markets, which is consistent with the common sense. For pre-commitment strategy, the amount invested in each risky asset in bull (bear) markets shows a decreasing (increasing) trend, and the amounts invested in different risky assets will become closer and closer. However, for the equilibrium strategy, the amount invested in each risky asset is relatively stable in every year. This indicates that the pre-commitment strategy is particularly sensitive to the market state at the initial moment, but the sensitiveness decreases gradually along with time. The sensitiveness of the equilibrium strategy to the market state is mostly the same in every year. The reason is that the pre-commitment strategy is made at the initial time, but the equilibrium strategy can be updated at the beginning of every year. Thus the stability of the equilibrium strategy is better than that of the pre-commitment strategy. But from Fig. 2, we find that the wealth accumulation under the equilibrium strategy is less than that under the pre-commitment strategy, that is, the equilibrium strategy is less efficient than the pre-commitment strategy.

The effect of the return of premiums clause on the pre-commitment strategy is shown in Fig. 3. We find that in bull (bear) markets, the amount invested in the risky assets (here and hereafter, we define the sum of the amount invested in the three risky assets as the amount invested in the risky assets) when the return of premiums clause is considered is less (more) than that when the return of premiums clause is not considered. However, in bear markets, the amount invested in the risky assets is less than zero, which means that the pension manager short sells the risky assets to buy the risk-free asset. Generally, the return of premiums clause decreases the transaction amount of the risky assets no matter in bull markets or in bear markets. One possible explanation is that the manager with the clause will face more uncertainty of the wealth in the future due to the return of premiums to the members who die before retirement, which in turn forces her to invest less in the risky assets to avoid higher risk. Moreover, we find that the effect of the return of premiums clause on the pre-commitment strategy grows larger with time. This is because both the mortality rate and the contribution accumulation increase with time, and hence the amount of the premiums returned to the dead members increases with time.

Fig. 4 shows the effect of regime switching on the two investment strategies. We find that in the case with regime switching, no matter for the pre-commitment or the equilibrium strategy, the amount invested in risky assets is positive in bull markets, which means the pension manager buys the risky assets; while it is negative in bear markets, which means the pension manager short



(a).



(b).

Fig. 4. The effect of regime switching on the two strategies.

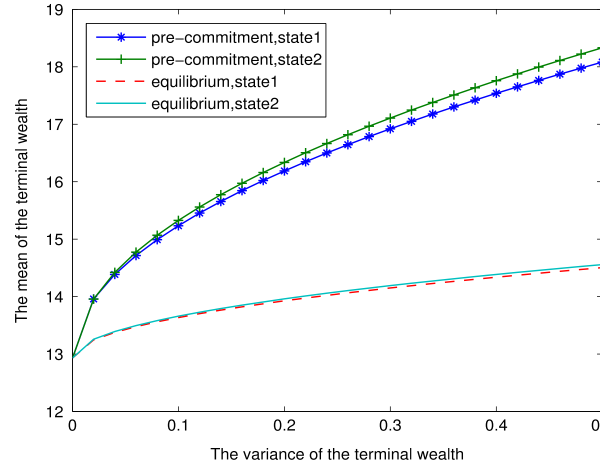


Fig. 5. Efficient frontiers with different initial market states.

sells the risky assets to buy the risk-free asset. In both states, the transaction amount of the risky assets is greater than that in the case with no regime switching. This is because regime switching can describe the real market better, and hence the manager obtains more information from the market, which can be served as her investment guidance. Moreover, the amount invested in risky assets is relatively stable in the case with no regime switching, while it fluctuates along with the market states in the case with regime switching. This is because, when regime switching is considered, the change of market states promotes the pension manager to adjust her investment strategies, which is more in line with the reality and makes our strategy more practical.

6.2. Numerical analysis for the two efficient frontiers

In this subsection, we consider the effect of regime switching and the return of premiums clause on the two efficient frontiers.

In order to study the effect of the market states on the efficient frontier, we plot in Fig. 5 these efficient frontiers with different initial market states for the pre-commitment or the equilibrium strategy, respectively. We find that no matter for the pre-commitment or the equilibrium strategy, the corresponding efficient frontier of state 2 lies above that of state 1. That is, for the same expected terminal wealth, the investor would bear less risk when she enters the market at bullish time. On the other hand, the efficient frontier with pre-commitment strategy lies above the efficient frontier with equilibrium strategy. That is, to obtain the same expected terminal wealth the investor with equilibrium strategy needs to face more risks than the investor with pre-commitment strategy. The reason is that the pre-commitment investor focuses on the globally optimal strategy, but the equilibrium investor takes the non-cooperative game to obtain the time-consistent strategy, which leads to the increase of the risk.

Under the assumption that the initial market state is bullish, Fig. 6 plots the effect of the return of premiums clause on the two efficient frontiers. We find that no matter for the pre-commitment or the equilibrium strategy, the efficient frontier with the return of premiums clause is below that without the return of premiums clause. That is, to get the same expected terminal wealth, greater risk will be faced if the return of premium clause is taken into consideration. The reason is that the manager of the DC pension fund with the clause needs to allocate part of the accumulated wealth to the members who die during the accumulation phase, which decreases the wealth level and increases the uncertainty of the wealth, that is, the manager with the clause needs to bear greater risk.

Fig. 7 plots the effect of regime switching on the two efficient frontiers. We find that no matter for the pre-commitment or the equilibrium strategy, the efficient frontier without regime switching is below that with regime switching. That is, to get the same expected terminal wealth, less risk will be faced if regime switching is taken into consideration. This shows that the investment risk can be better avoided when regime switching is considered.

7. Conclusion

This paper studies the pre-commitment and equilibrium strategies for a DC pension plan under a multi-period mean–variance framework. In the plan, the pension manager is assumed to invest one risk-free asset and n risky assets in a financial market. In order to protect the interests of pension members who die before retirement, we introduce the return of premiums clause to the model. Moreover, we use the Markov regime switching model to show the effect of market states on the return of the risky assets.

On the one hand, using the embedding technique and the dynamic programming method, we obtain the pre-commitment strategy and the corresponding efficient frontier in closed form. We find that the pre-commitment strategy depends not only on

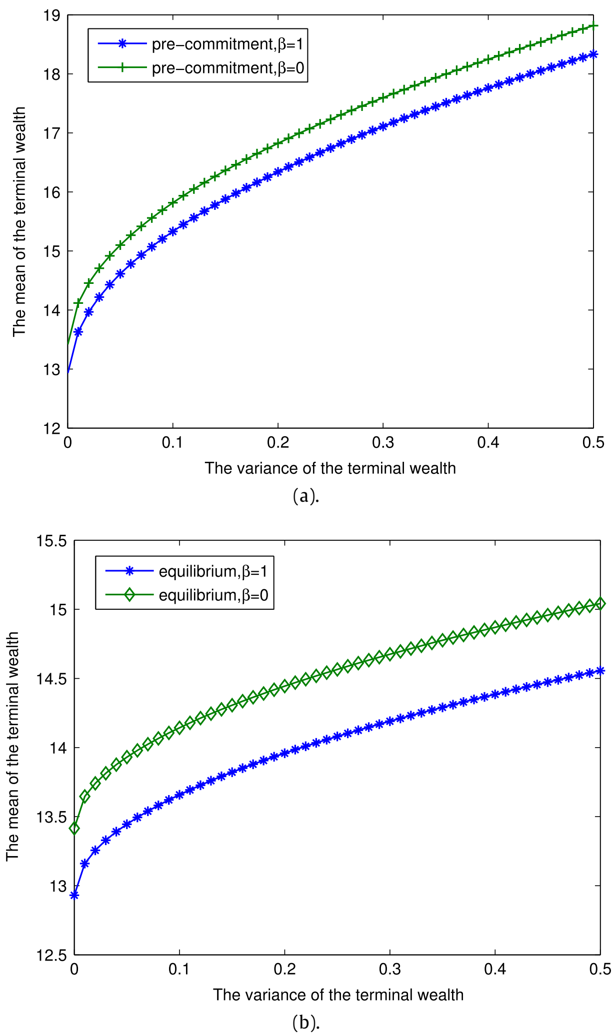


Fig. 6. The effect of the return of premiums clause on the two efficient frontiers.

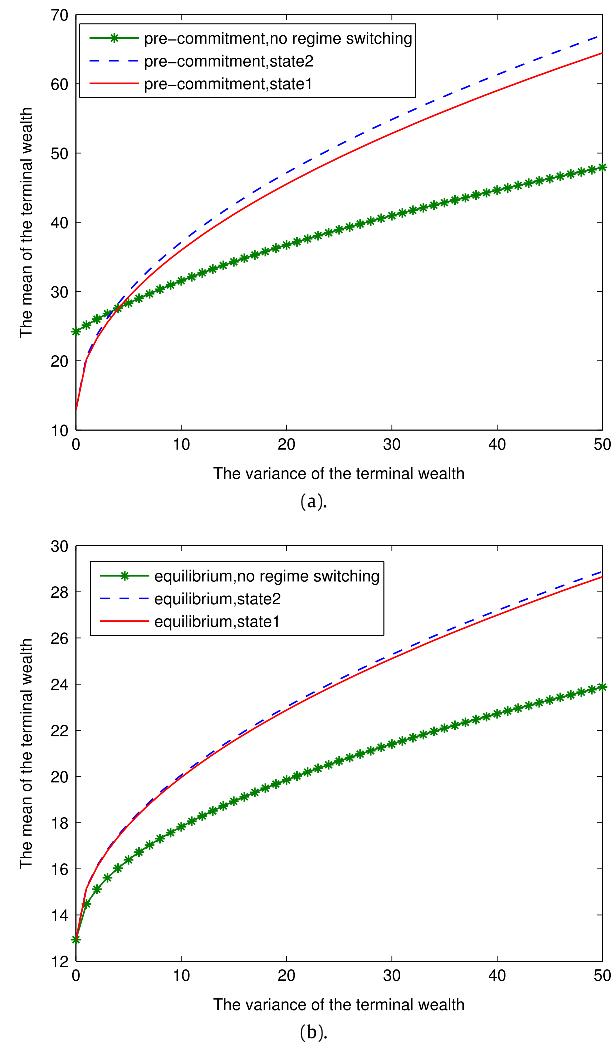


Fig. 7. The effect of regime switching on the two efficient frontiers.

the current state and wealth but also on the initial state and wealth. Moreover, the pre-commitment strategy depends on the contribution, the survival rate, the interest rate and the return of premiums clause.

On the other hand, using the game theory and the extended Bellman equation, we derive the analytical expressions of the equilibrium strategy and the corresponding efficient frontier. We find that the equilibrium strategy is only dependent of the current state, the future interest rate and survival rate, but independent of other factors. This is quite different from the pre-commitment strategy.

Finally, we do some numerical analyses for the two strategies and efficient frontiers and find some interesting results:

(i) The return of premiums clause decreases the amount invested in the risky assets in the pre-commitment strategy, but has no effect on the equilibrium strategy.

(ii) In both strategies, regime switching increases the transaction amount of the risky assets, and the strategies are more practical when regime switching is considered.

(iii) In both strategies, to obtain the same expected terminal wealth, the manager faces greater risk when the return of premiums clause is considered, but faces less risk when regime switching is considered.

(iv) To obtain the same expected terminal wealth, the equilibrium investor needs to face more risks than the pre-commitment investor.

Our work can be extended to several directions. For example, we can further consider the stochastic salary flow and inflation risk. We can also add the consumption problem into our model to consider the life cycle problem. Another interesting topic is to consider our model under incomplete information. In our model we assume that the states of the financial market are fully observable. In practice, the states of the financial market cannot be completely observed, see Zhang et al. (2016).

Acknowledgments

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Appendix A. The proof of Lemma 3.2

Proof. We prove the expression of M_k by mathematical induction. For $k = T$, we have

$$M_T = \lambda A_{T,T-1} - 2\omega \sum_{l=T}^{T-1} B_l(\beta) A_{k,l} A^2_{l+1,T-1} = \lambda.$$

That is, Eq. (16) holds for $k = T$.

Suppose that Eq. (16) holds for $T, T-1, \ldots, k+1$. Then we have

$$\begin{aligned} M_k &= \left(M_{k+1} - 2\omega A^2_{k+1,T-1} B_k(\beta)\right) A_{k,k} \\ &= \left(\lambda A_{k+1,T-1} - 2\omega \sum_{l=k+1}^{T-1} B_l(\beta) A_{k+1,l} A^2_{l+1,T-1} - 2\omega A^2_{k+1,T-1} B_k(\beta)\right) A_{k,k} \\ &= \lambda A_{k,T-1} - 2\omega \sum_{l=k+1}^{T-1} B_l(\beta) A_{k,l} A^2_{l+1,T-1} - 2\omega B_k(\beta) A_{k,k} A^2_{k+1,T-1} \\ &= \lambda A_{k,T-1} - 2\omega \sum_{l=k}^{T-1} B_l(\beta) A_{k,l} A^2_{l+1,T-1}, \end{aligned} \quad (75)$$

which means that Eq. (16) holds for k. By the principle of mathematical induction, Eq. (16) holds for all $k = 0, 1, \ldots, T$, and thus the lemma is proved. □

Appendix B. The proof of Lemma 3.3

Proof. First, we prove the expression of $\boldsymbol{\eta}_{\mathbf{k}}$ by mathematical induction. For $k = T$, we have

$$\boldsymbol{\eta}_{\mathbf{T}} = \left(\prod_{m=T}^{T-1} \mathbf{f_m} Q(m)\right) \mathbb{I} = \mathbb{I},$$

implying that Eq. (19) holds for $k = T$.

Suppose that Eq. (19) holds for $T, T-1, \ldots, k+1$. Then we have

$$\boldsymbol{\eta}_{\mathbf{k}} = \mathbf{f_k} Q(k) \boldsymbol{\eta}_{\mathbf{k+1}} = \mathbf{f_k} Q(k) \left(\prod_{m=k+1}^{T-1} \mathbf{f_m} Q(m)\right) \mathbb{I} = \left(\prod_{m=k}^{T-1} \mathbf{f_m} Q(m)\right) \mathbb{I}, \quad (76)$$

which means that Eq. (19) holds for k. Hence Eq. (19) holds for all $k = 0, 1, \ldots, T$.

Next, we prove the expression of $\mathbf{D_k}$. For $k = T$, we have

$$\begin{aligned} \mathbf{D_T} &= \sum_{m=T}^{T-1} \frac{M^2_{m+1}}{4\omega A^2_{m+1,T-1}} \left(\prod_{l=T}^{m-1} Q(l)\right) \mathbf{h_m} Q(m) \boldsymbol{\eta}_{\mathbf{m+1}} \\ &+ \sum_{m=T}^{T-1} \left(M_{m+1} B_m(\beta) - \omega A^2_{m+1,T-1} B^2_m(\beta)\right) \left(\prod_{l=T}^{m-1} Q(l)\right) \boldsymbol{\eta}_{\mathbf{m}} = \mathbf{0}, \end{aligned}$$

that is, Eq. (20) holds for $k = T$.

Suppose that Eq. (20) holds for $T, T-1, \ldots, k+1$. Then we have

$$\begin{aligned} \mathbf{D_k} &= \frac{M^2_{k+1}}{4\omega A^2_{k+1,T-1}} \mathbf{h_k} Q(k) \boldsymbol{\eta}_{\mathbf{k+1}} + Q(k) \mathbf{D_{k+1}} \\ &+ \left(M_{k+1} B_k(\beta) - \omega A^2_{k+1,T-1} B^2_k(\beta)\right) \boldsymbol{\eta}_{\mathbf{k}} \\ &= \frac{M^2_{k+1}}{4\omega A^2_{k+1,T-1}} \mathbf{h_k} Q(k) \boldsymbol{\eta}_{\mathbf{k+1}} + \left(M_{k+1} B_k(\beta) - \omega A^2_{k+1,T-1} B^2_k(\beta)\right) \boldsymbol{\eta}_{\mathbf{k}} \\ &+ Q(k) \left(\sum_{m=k+1}^{T-1} \frac{M^2_{m+1}}{4\omega A^2_{m+1,T-1}} \left(\prod_{l=k+1}^{m-1} Q(l)\right) \mathbf{h_m} Q(m) \boldsymbol{\eta}_{\mathbf{m+1}} \right. \\ &+ \left. \sum_{m=k+1}^{T-1} \left(M_{m+1} B_m(\beta) - \omega A^2_{m+1,T-1} B^2_m(\beta)\right) \left(\prod_{l=k+1}^{m-1} Q(l)\right) \boldsymbol{\eta}_{\mathbf{m}} \right) \\ &= \sum_{m=k}^{T-1} \frac{M^2_{m+1}}{4\omega A^2_{m+1,T-1}} \left(\prod_{l=k}^{m-1} Q(l)\right) \mathbf{h_m} Q(m) \boldsymbol{\eta}_{\mathbf{m+1}} \\ &+ \sum_{m=k}^{T-1} \left(M_{m+1} B_m(\beta) - \omega A^2_{m+1,T-1} B^2_m(\beta)\right) \left(\prod_{l=k}^{m-1} Q(l)\right) \boldsymbol{\eta}_{\mathbf{m}}, \end{aligned} \quad (77)$$

which shows that Eq. (20) holds for k. By the principle of mathematical induction, Eq. (20) holds for all $k = 0, 1, \ldots, T$. The proof is completed. □

Appendix C. The proof of Lemma 3.4

Proof. We prove this lemma by mathematical induction for k. For $k = T-1$ and $i \in \Pi$, Lemma 3.3 implies that

$$\eta_{T-1}(i) = \left(\mathbf{f_{T-1}} Q(T-1) \mathbb{I}\right)(i) = f_{T-1}(i).$$

From Remark 3.1, $0 < f_k(i) < 1$ for $k = 0, 1, \ldots, T-1$ and $i \in \Pi$. Hence, Eq. (21) holds for $k = T-1$ and all $i \in \Pi$.

Suppose that Eq. (21) holds for $T-1, T-2, \ldots, k+1$ and all $i \in \Pi$. Set $\bar{\eta}_{k+1} = \max_{j \in \Pi}\{\eta_{k+1}(j)\}$ and $\tilde{\eta}_{k+1} = \min_{j \in \Pi}\{\eta_{k+1}(j)\}$. Then,

$$0 < \tilde{\eta}_{k+1} \le \sum_{j=1}^{J} q_{ij}(k) \eta_{k+1}(j) \le \bar{\eta}_{k+1} < 1. \quad (78)$$

Again from the fact $0 < f_k(i) < 1$ and Eqs. (14) and (78), we have $0 < \eta_k(i) < 1$ for all $i \in \Pi$, i.e., Eq. (21) holds for k and all $i \in \Pi$. This completes the proof. □

Appendix D. The proof of Theorem 3.5

Proof. For $k = T-1$, by Eqs. (8) and (9), we have

$$\begin{aligned} & v_{T-1}(x_{T-1}, i) \\ &= \max_{\pi_{T-1}(i)} \left\{ \sum_{j=1}^{J} q_{ij}(T-1) \mathbb{E}\left[v_T \left(A_{T-1,T-1} x_{T-1} + B_{T-1}(\beta) + \frac{S'_{T-1}(i) \pi_{T-1}(i)}{p_{T-1+y}}, j \right) \right] \right\} \\ &= \max_{\pi_{T-1}(i)} \left\{ \sum_{j=1}^{J} q_{ij}(T-1) \mathbb{E}\left[-\omega \left(A_{T-1,T-1} x_{T-1} + B_{T-1}(\beta) + \frac{S'_{T-1}(i) \pi_{T-1}(i)}{p_{T-1+y}} \right)^2 \right. \right. \\ &\quad \left. \left. + \lambda \left(A_{T-1,T-1} x_{T-1} + B_{T-1}(\beta) + \frac{S'_{T-1}(i) \pi_{T-1}(i)}{p_{T-1+y}} \right) \right] \right\} \\ &= -\omega \left(A_{T-1,T-1} x_{T-1} + B_{T-1}(\beta) \right)^2 + \lambda \left(A_{T-1,T-1} x_{T-1} + B_{T-1}(\beta) \right) \\ &\quad + \max_{\pi_{T-1}(i)} \left\{ \left[\lambda - 2\omega \left(A_{T-1,T-1} x_{T-1} + B_{T-1}(\beta) \right) \right] \frac{S'_{T-1}(i) \pi_{T-1}(i)}{p_{T-1+y}} \right. \\ &\quad \left. - \omega \frac{\pi'_{T-1}(i) \Upsilon_{T-1}(i) \pi_{T-1}(i)}{p^2_{T-1+y}} \right\}. \end{aligned} \quad (79)$$

Since $\Upsilon_{T-1}(i)\,(i \in \Pi)$ is positive definite and $\omega > 0$, the application of the first order condition to $\pi_{T-1}(i)$ yields the optimal solution

$$\hat{\pi}^A_{T-1}(i) = \left(\frac{\lambda}{2\omega} - A_{T-1,T-1} x_{T-1} - B_{T-1}(\beta) \right) p_{T-1+y} \Upsilon^{-1}_{T-1}(i) s_{T-1}(i). \quad (80)$$

Substituting Eq. (80) into Eq. (79) and noticing that $\eta_{T-1}(i) = f_{T-1}(i)$, we have

$$\begin{aligned} & v_{T-1}\left(x_{T-1}, i\right) \\ &= -\omega \left(A_{T-1,T-1} x_{T-1} + B_{T-1}(\beta) \right)^2 f_{T-1}(i) \\ &\quad + \lambda \left(A_{T-1,T-1} x_{T-1} + B_{T-1}(\beta) \right) f_{T-1}(i) + \frac{\lambda^2}{4\omega} h_{T-1}(i) \\ &= -\omega A^2_{T-1,T-1} f_{T-1}(i) x^2_{T-1} + \left(\lambda - 2\omega B_{T-1}(\beta) \right) A_{T-1,T-1} f_{T-1}(i) x_{T-1} \\ &\quad + \frac{\lambda^2}{4\omega} h_{T-1}(i) + \left(\lambda B_{T-1}(\beta) - \omega B^2_{T-1}(\beta) \right) f_{T-1}(i) \\ &= -\omega A^2_{T-1,T-1} \eta_{T-1}(i) x^2_{T-1} + M_{T-1} \eta_{T-1}(i) x_{T-1} + D_{T-1}(i). \end{aligned} \quad (81)$$

Eqs. (80) and (81) show that Eqs. (22) and (23) hold for $k = T-1$.

Suppose that Eqs. (22) and (23) hold for $T-1, T-2, \ldots, k+1$. Substituting the expression of v_{k+1} in Eq. (22) into Eq. (8), we have

$$\begin{aligned}
&v_k(x_k, i)\\
&= \max_{\pi_k(i)}\left\{\sum_{j=1}^{J} q_{ij}(k)\mathbb{E}\left[v_{k+1}\left(A_{k,k}x_k + B_k(\beta) + \frac{S_k'(i)\pi_k(i)}{p_{k+y}}, j\right)\right]\right\}\\
&= \max_{\pi_k(i)}\left\{\sum_{j=1}^{J} q_{ij}(k)\mathbb{E}\left[-\omega A^2_{k+1,T-1}\eta_{k+1}(j)\left(A_{k,k}x_k + B_k(\beta) + \frac{S_k'(i)\pi_k(i)}{p_{k+y}}\right)^2\right.\right.\\
&\quad \left.\left. + M_{k+1}\eta_{k+1}(j)\left(A_{k,k}x_k + B_k(\beta) + \frac{S_k'(i)\pi_k(i)}{p_{k+y}}\right) + D_{k+1}(j)\right]\right\}\\
&= \sum_{j=1}^{J} q_{ij}(k)\left(-\omega A^2_{k+1,T-1}\eta_{k+1}(j)(A_{k,k}x_k + B_k(\beta))^2\right.\\
&\quad \left. + M_{k+1}\eta_{k+1}(j)\left(A_{k,k}x_k + B_k(\beta)\right) + D_{k+1}(j)\right)\\
&\quad + \max_{\pi_k(i)}\left\{\sum_{j=1}^{J} q_{ij}(k)\eta_{k+1}(j)\left[\left(-2\omega A^2_{k+1,T-1}\left(A_{k,k}x_k + B_k(\beta)\right)\right.\right.\right.\\
&\quad \left.\left.\left. + M_{k+1}\right)\frac{s_k'(i)\pi_k(i)}{p_{k+y}} - \omega A^2_{k+1,T-1}\frac{\pi_k'(i)\Upsilon_k(i)\pi_k(i)}{p^2_{k+y}}\right]\right\}. \qquad (82)
\end{aligned}$$

Since $A^2_{k+1,T-1} > 0$, $\omega > 0$, $\Upsilon_k(i)$ is positive definite, and $0 < \sum_{j=1}^{J} q_{ij}(k)\eta_{k+1}(j) < 1$ $(i \in \Pi)$ by Lemma 3.4, the application of the first order condition to $\pi_k(i)$ yields the optimal solution

$$\begin{aligned}
\hat{\pi}_k^A(i) &= \left(\frac{M_{k+1}}{2\omega A^2_{k+1,T-1}} - A_{k,k}x_k - B_k(\beta)\right) p_{k+y}\Upsilon_k^{-1}(i)s_k(i)\\
&= \left(-\sum_{l=k}^{T-1}\frac{B_l(\beta)}{A_{k+1,l}} + \frac{\lambda}{2\omega A_{k+1,T-1}} - A_{k,k}x_k\right) p_{k+y}\Upsilon_k^{-1}(i)s_k(i). \qquad (83)
\end{aligned}$$

Substituting Eq. (83) into Eq. (82) gives

$$\begin{aligned}
&v_k(x_k, i)\\
&= -\omega A^2_{k+1,T-1}\left(A_{k,k}x_k + B_k(\beta)\right)^2 f_k(i)\sum_{j=1}^{J} q_{ij}(k)\eta_{k+1}(j)\\
&\quad + \frac{M^2_{k+1}}{4\omega A^2_{k+1,T-1}} h_k(i)\sum_{j=1}^{J} q_{ij}(k)\eta_{k+1}(j)\\
&\quad + M_{k+1}\left(A_{k,k}x_k + B_k(\beta)\right) f_k(i)\sum_{j=1}^{J} q_{ij}(k)\eta_{k+1}(j) + \sum_{j=1}^{J} q_{ij}(k)D_{k+1}(j)\\
&= -\omega A^2_{k+1,T-1}A^2_{k,k}\eta_k(i)x_k^2 + \left(M_{k+1} - 2\omega A^2_{k+1,T-1}B_k(\beta)\right) A_{k,k}\eta_k(i)x_k\\
&\quad + \frac{M^2_{k+1}}{4\omega A^2_{k+1,T-1}} h_k(i)\sum_{j=1}^{J} q_{ij}(k)\eta_{k+1}(j)\\
&\quad + \left(M_{k+1}B_k(\beta) - \omega A^2_{k+1,T-1}B_k^2(\beta)\right)\eta_k(i) + \sum_{j=1}^{J} q_{ij}(k)D_{k+1}(j)\\
&= -\omega A^2_{k,T-1}\eta_k(i)x_k^2 + M_k\eta_k(i)x_k + D_k(i). \qquad (84)
\end{aligned}$$

Eqs. (83) and (84) show that Eqs. (22) and (23) hold for k. By the principle of mathematical induction, Eqs. (22) and (23) hold for all $k = 0, 1, \ldots, T-1$. □

Appendix E. The proof of Lemma 3.6

Proof. First, we prove the following equation by mathematical induction: for $k = 0, 1, \ldots, T-1$,

$$\mathbb{E}\left(\prod_{l=k}^{T-1} f_l(\xi_l)|\xi_k = i_k\right) = \eta_k(i_k). \qquad (85)$$

For $k = T-1$,

$$\begin{aligned}
&\mathbb{E}\left(\prod_{l=T-1}^{T-1} f_l(\xi_l)|\xi_{T-1} = i_{T-1}\right) = f_{T-1}(i_{T-1})\\
&= \left(\mathbf{f_{T-1}}Q(T-1)\mathbb{I}\right)(i_{T-1}) = \eta_{T-1}(i_{T-1}).
\end{aligned} \qquad (86)$$

Hence, Eq. (85) holds for $k = T-1$. Suppose that Eq. (85) holds for $T-1, T-2, \ldots, k+1$. Then, we have

$$\begin{aligned}
&\mathbb{E}\left(\prod_{l=k}^{T-1} f_l(\xi_l)|\xi_k = i_k\right) = \mathbb{E}\left(f_k(\xi_k)\prod_{l=k+1}^{T-1} f_l(\xi_l)|\xi_k = i_k\right)\\
&= f_k(i_k)\mathbb{E}\left(\prod_{l=k+1}^{T-1} f_l(\xi_l)|\xi_k = i_k\right) = f_k(i_k)\mathbb{E}\left(\mathbb{E}\left(\prod_{l=k+1}^{T-1} f_l(\xi_l)|\xi_{k+1}\right)|\xi_k = i_k\right)\\
&= f_k(i_k)\mathbb{E}\left(\eta_{k+1}(\xi_{k+1})|\xi_k = i_k\right) = f_k(i_k)\sum_{j=1}^{J} q_{i_kj}(k)\eta_{k+1}(j) = \eta_k(i_k), \qquad (87)
\end{aligned}$$

which shows that Eq. (85) holds for k. By the principle of mathematical induction, Eq. (85) holds for all $k = 0, 1, \ldots, T-1$.

Next, we prove Eq. (31) by mathematical induction. For $t = k$, Eq. (85) gives

$$\theta_k(i_k) = \mathbb{E}\left(\prod_{l=k}^{T-1} f_l(\xi_l)|\xi_k = i_k\right) = \eta_k(i_k) = \left(\left(\prod_{l=k}^{k-1} Q(l)\right)\boldsymbol{\eta_k}\right)(i_k),$$

which means that Eq. (31) holds for $t = k$. Suppose that Eq. (31) holds for $k, k-1, \ldots, t+1$. Then we have

$$\begin{aligned}
\theta_k(i_t) &= \mathbb{E}\left(\prod_{l=k}^{T-1} f_l(\xi_l)|\xi_t = i_t\right) = \mathbb{E}\left(\mathbb{E}\left(\prod_{l=k}^{T-1} f_l(\xi_l)|\xi_{t+1}\right)|\xi_t = i_t\right)\\
&= \mathbb{E}\left(\theta_k(\xi_{t+1})|\xi_t = i_t\right) = \mathbb{E}\left(\left(\left(\prod_{l=t+1}^{k-1} Q(l)\right)\boldsymbol{\eta_k}\right)(\xi_{t+1})|\xi_t = i_t\right)\\
&= \sum_{j=1}^{J} q_{i_tj}(t)\left(\left(\prod_{l=t+1}^{k-1} Q(l)\right)\boldsymbol{\eta_k}\right)(j) = \left(\left(\prod_{l=t}^{k-1} Q(l)\right)\boldsymbol{\eta_k}\right)(i_t), \qquad (88)
\end{aligned}$$

which shows that Eq. (31) holds for t. Hence, Eq. (31) holds for all $t = 0, 1, \ldots, k$.

Last, the proof of Eq. (32) is similar to that of Eq. (31) and is omitted. □

Appendix F. The proof of Lemma 3.9

Proof. For $k = 0, 1, \ldots, T-1$, $\xi_0 = i_0 \in \Pi$, by Lemma 3.8, we have

$$\begin{aligned}
b_k(\beta, i_0) &= \sum_{l=0}^{k}\theta_l(i_0)\chi_l(\beta) - \sum_{l=0}^{k}\phi_l(i_0)\sum_{m=l+1}^{T-1}\chi_m(\beta)\\
&= \sum_{l=0}^{k}\theta_l(i_0)\chi_l(\beta) - \sum_{l=0}^{k}\theta_{l+1}(i_0)\sum_{m=l+1}^{T-1}\chi_m(\beta) + \sum_{l=0}^{k}\theta_l(i_0)\sum_{m=l+1}^{T-1}\chi_m(\beta)\\
&= \sum_{l=0}^{k}\theta_l(i_0)\sum_{m=l}^{T-1}\chi_m(\beta) - \sum_{l=0}^{k}\theta_{l+1}(i_0)\sum_{m=l+1}^{T-1}\chi_m(\beta)\\
&= \theta_0(i_0)\sum_{m=0}^{T-1}\chi_m(\beta) - \theta_{k+1}(i_0)\sum_{m=k+1}^{T-1}\chi_m(\beta). \qquad (89)
\end{aligned}$$

This completes the proof. □

Appendix G. The proof of Theorem 3.10

Proof. Substituting Eq. (23) into wealth process (1), we have

$$X_{k+1}^{\hat{\pi}^A} = A_{k,k}X_k^{\hat{\pi}^A} + B_k(\beta)$$

$$+\left(\frac{M_{k+1}}{2\omega A^2_{k+1,T-1}}-A_{k,k}X_k^{\hat{\pi}^A}-B_k(\beta)\right)S'_k(\xi_k)\Upsilon_k^{-1}(\xi_k)s_k(\xi_k)$$
$$=\left(A_{k,k}X_k^{\hat{\pi}^A}+B_k(\beta)\right)\left(1-S'_k(\xi_k)\Upsilon_k^{-1}(\xi_k)s_k(\xi_k)\right)$$
$$+\left(\frac{M_{k+1}}{2\omega A^2_{k+1,T-1}}\right)S'_k(\xi_k)\Upsilon_k^{-1}(\xi_k)s_k(\xi_k). \quad (90)$$

Noticing that $X_k^{\hat{\pi}^A}$ and $S'_k(\xi_k)$ are statistical independent, taking the conditional expectation on both sides of Eq. (90) under $\xi_0, \xi_1, \ldots, \xi_k$ yields

$$\mathbb{E}\left(X_{k+1}^{\hat{\pi}^A}|\xi_0,\xi_1,\ldots,\xi_k\right)=A_{k,k}f_k(\xi_k)\mathbb{E}\left(X_k^{\hat{\pi}^A}|\xi_0,\xi_1,\ldots,\xi_k\right)$$
$$+B_k(\beta)f_k(\xi_k)+\frac{M_{k+1}}{2\omega A^2_{k+1,T-1}}h_k(\xi_k). \quad (91)$$

Since $\mathbb{E}(.|\xi_0,\xi_1,\ldots,\xi_k) = \mathbb{E}(.|\xi_0,\xi_1,\ldots,\xi_{k-1})$, by applying Eq. (91) recursively, we can obtain

$$\mathbb{E}\left(X_k^{\hat{\pi}^A}|\xi_0,\xi_1,\ldots,\xi_{k-1},X_0\right)$$
$$=A_{0,k-1}\prod_{l=0}^{k-1}f_l(\xi_l)X_0+\sum_{l=0}^{k-1}B_l(\beta)A_{l+1,k-1}\prod_{m=l}^{k-1}f_m(\xi_m)$$
$$+\sum_{l=0}^{k-1}\frac{M_{l+1}}{2\omega A^2_{l+1,T-1}}A_{l+1,k-1}h_l(\xi_l)\prod_{m=l+1}^{k-1}f_m(\xi_m). \quad (92)$$

Then, at time T, we have

$$\mathbb{E}\left(X_T^{\hat{\pi}^A}|\xi_0,\xi_1,\ldots,\xi_{T-1},X_0\right)$$
$$=A_{0,T-1}\prod_{l=0}^{T-1}f_l(\xi_l)X_0+\sum_{l=0}^{T-1}B_l(\beta)A_{l+1,T-1}\prod_{m=l}^{T-1}f_m(\xi_m)$$
$$+\sum_{l=0}^{T-1}\frac{M_{l+1}}{2\omega A_{l+1,T-1}}h_l(\xi_l)\prod_{m=l+1}^{T-1}f_m(\xi_m). \quad (93)$$

Again, taking the conditional expectation on both sides of Eq. (93) under the initial state $\xi_0 = i_0$ and the initial wealth $X_0 = x_0$, we have

$$\mathbb{E}_{0,x_0,i_0}\left(X_T^{\hat{\pi}^A}\right)$$
$$=A_{0,T-1}\mathbb{E}\left(\prod_{l=0}^{T-1}f_l(\xi_l)|\xi_0=i_0\right)x_0$$
$$+\sum_{l=0}^{T-1}B_l(\beta)A_{l+1,T-1}\mathbb{E}\left(\prod_{m=l}^{T-1}f_m(\xi_m)|\xi_0=i_0\right)$$
$$+\sum_{l=0}^{T-1}\frac{M_{l+1}}{2\omega A_{l+1,T-1}}\mathbb{E}\left(h_l(\xi_l)\prod_{m=l+1}^{T-1}f_m(\xi_m)|\xi_0=i_0\right)$$
$$=A_{0,T-1}\theta_0(i_0)x_0+\sum_{l=0}^{T-1}B_l(\beta)A_{l+1,T-1}\theta_l(i_0)+\sum_{l=0}^{T-1}\frac{M_{l+1}}{2\omega A_{l+1,T-1}}\phi_l(i_0)$$
$$=A_{0,T-1}\theta_0(i_0)x_0+\sum_{l=0}^{T-1}B_l(\beta)A_{l+1,T-1}\theta_l(i_0)$$
$$-\sum_{l=0}^{T-1}\phi_l(i_0)\sum_{m=l+1}^{T-1}B_m(\beta)A_{m+1,T-1}+\frac{\lambda}{2\omega}\sum_{l=0}^{T-1}\phi_l(i_0)$$
$$=A_{0,T-1}\theta_0(i_0)x_0+b_{T-1}(\beta,i_0)+\frac{\lambda}{2\omega}a_0(i_0). \quad (94)$$

This shows Eq. (33). In order to derive $\mathbb{E}_{0,x_0,i_0}\left(\left(X_T^{\hat{\pi}^A}\right)^2\right)$, we first note that

$$\mathbb{E}\left[(S'_k(\xi_k)\Upsilon_k^{-1}(\xi_k)s_k(\xi_k))^2\right]$$
$$=\mathbb{E}\left[s'_k(\xi_k)\Upsilon_k^{-1}(\xi_k)S_k(\xi_k)S'_k(\xi_k)\Upsilon_k^{-1}(\xi_k)s_k(\xi_k)\right]$$
$$=s'_k(\xi_k)\Upsilon_k^{-1}(\xi_k)s_k(\xi_k)=h_k(\xi_k). \quad (95)$$

Taking the square on both sides of Eq. (90) and then taking the conditional expectation, we get

$$\mathbb{E}\left(\left(X_{k+1}^{\hat{\pi}^A}\right)^2|\xi_0,\xi_1,\ldots,\xi_k\right)$$
$$=\mathbb{E}\left(\left(X_k^{\hat{\pi}^A}\right)^2|\xi_0,\xi_1,\ldots,\xi_k\right)A^2_{k,k}f_k(\xi_k)$$
$$+2\mathbb{E}\left(X_k^{\hat{\pi}^A}|\xi_0,\xi_1,\ldots,\xi_k\right)A_{k,k}B_k(\beta)f_k(\xi_k)$$
$$+\frac{M^2_{k+1}}{4\omega^2A^4_{k+1,T-1}}h_k(\xi_k)+B_k^2(\beta)f_k(\xi_k). \quad (96)$$

Using $\mathbb{E}(.|\xi_0,\xi_1,\ldots,\xi_k) = \mathbb{E}(.|\xi_0,\xi_1,\ldots,\xi_{k-1})$ again, by applying Eq. (96) recursively and substituting Eq. (92) and the expression of M_{k+1} into it, we can obtain

$$\mathbb{E}\left(\left(X_T^{\hat{\pi}^A}\right)^2|\xi_0,\xi_1,\ldots,\xi_{T-1},X_0\right)$$
$$=A^2_{0,T-1}\prod_{l=0}^{T-1}f_l(\xi_l)X_0^2+2A_{0,T-1}\sum_{l=0}^{T-1}\chi_l(\beta)\prod_{m=0}^{T-1}f_m(\xi_m)X_0$$
$$+\sum_{l=0}^{T-1}\chi_l^2(\beta)\prod_{m=l}^{T-1}f_m(\xi_m)$$
$$+2\sum_{l=0}^{T-1}\chi_l(\beta)\left(\sum_{m=0}^{l-1}\chi_m(\beta)\prod_{s=m}^{T-1}f_s(\xi_s)\right.$$
$$\left.-\sum_{m=0}^{l-1}\sum_{s=m+1}^{T-1}\chi_s(\beta)h_m(\xi_m)\prod_{s=m+1}^{T-1}f_s(\xi_s)\right)$$
$$+\frac{\lambda}{\omega}\sum_{l=0}^{T-1}\left(\chi_l(\beta)\sum_{m=0}^{l-1}h_m(\xi_m)\prod_{s=m+1}^{T-1}f_s(\xi_s)\right.$$
$$\left.-\sum_{m=l+1}^{T-1}\chi_m(\beta)h_l(\xi_l)\prod_{m=l+1}^{T-1}f_m(\xi_m)\right)$$
$$+\sum_{l=0}^{T-1}\left(\sum_{m=l+1}^{T-1}\chi_m(\beta)\right)^2h_l(\xi_l)\prod_{m=l+1}^{T-1}f_m(\xi_m)$$
$$+\frac{\lambda^2}{4\omega^2}\sum_{l=0}^{T-1}h_l(\xi_l)\prod_{m=l+1}^{T-1}f_m(\xi_m). \quad (97)$$

Once more, taking the conditional expectation on both sides of Eq. (97) under the initial state $\xi_0 = i_0$ and the initial wealth $X_0 = x_0$, we obtain

$$\mathbb{E}_{0,x_0,i_0}\left(\left(X_T^{\hat{\pi}^A}\right)^2\right)$$
$$=A^2_{0,T-1}\theta_0(i_0)x_0^2+2A_{0,T-1}\sum_{l=0}^{T-1}\chi_l(\beta)\theta_0(i_0)x_0$$
$$+\sum_{l=0}^{T-1}\chi_l^2(\beta)\theta_l(i_0)+\frac{\lambda^2}{4\omega^2}\sum_{l=0}^{T-1}\phi_l(i_0)$$

$$+\frac{\lambda}{\omega}\sum_{l=0}^{T-1}\left(\chi_l(\beta)\sum_{m=0}^{l-1}\phi_m(i_0)-\phi_l(i_0)\sum_{m=l+1}^{T-1}\chi_m(\beta)\right)$$
$$+\sum_{l=0}^{T-1}\left(\sum_{m=l+1}^{T-1}\chi_m(\beta)\right)^2\phi_l(i_0)$$
$$+2\sum_{l=0}^{T-1}\chi_l(\beta)\left(\sum_{m=0}^{l-1}\chi_m(\beta)\theta_m(i_0)-\sum_{m=0}^{l-1}\phi_m(i_0)\sum_{s=m+1}^{T-1}\chi_s(\beta)\right)$$
$$=A_{0,T-1}^2\theta_0(i_0)x_0^2+2\zeta(\beta)\theta_0(i_0)x_0+\frac{\lambda^2}{4\omega^2}a_0(i_0)+\psi(\beta,i_0), \tag{98}$$

where the last equality uses the following conclusion:

$$\sum_{l=0}^{T-1}\left(\chi_l(\beta)\sum_{m=0}^{l-1}\phi_m(i_0)-\phi_l(i_0)\sum_{m=l+1}^{T-1}\chi_m(\beta)\right)$$
$$=\sum_{l=0}^{T-1}\left(\chi_l(\beta)(\theta_l(i_0)-\theta_0(i_0))-(\theta_{l+1}(i_0)-\theta_l(i_0))\sum_{m=l+1}^{T-1}\chi_m(\beta)\right)$$
$$=\sum_{l=0}^{T-1}\theta_l(i_0)\sum_{m=l}^{T-1}\chi_m(\beta)-\sum_{l=0}^{T-1}\theta_{l+1}(i_0)\sum_{m=l+1}^{T-1}\chi_m(\beta)-\theta_0(i_0)\sum_{l=0}^{T-1}\chi_l(\beta)$$
$$=\theta_0(i_0)\sum_{m=0}^{T-1}\chi_m(\beta)-\theta_0(i_0)\sum_{l=0}^{T-1}\chi_l(\beta)=0.$$

This proves Eq. (34). □

Appendix H. The proof of Theorem 3.11

Proof. As pointed out earlier, the optimal strategy of problem $P(\omega)$ is the solution of $A(\lambda,\omega)$ with $\lambda=1+2\omega\mathbb{E}_{0,x_0,i_0}\left(X_T^{\hat{\pi}^A}\right)$. By Theorem 3.10, we have

$$\lambda=1+2\omega\mathbb{E}_{0,x_0,i_0}\left(X_T^{\hat{\pi}^A}\right)=1+2\omega\left(A_{0,T-1}\theta_0(i_0)x_0+b_{T-1}(\beta,i_0)\right)+\lambda a_0(i_0).$$

Because $0<1-a_0(i_0)=\theta_0(i_0)=\eta_0(i_0)<1$, the above equation gives

$$\lambda=\frac{1+2\omega\left(A_{0,T-1}\eta_0(i_0)x_0+b_{T-1}(\beta,i_0)\right)}{\eta_0(i_0)}. \tag{99}$$

Substituting Eq. (99) into Eq. (23), we obtain

$$\hat{\pi}_k^P(i)=\left(-\sum_{l=k}^{T-1}\frac{B_l(\beta)}{A_{k+1,l}}+\frac{1+2\omega b_{T-1}(\beta,i_0)}{2\omega\eta_0(i_0)A_{k+1,T-1}}+A_{0,k}x_0-A_{k,k}x_k\right)\times p_{k+y}\Upsilon_k^{-1}(i)s_k(i). \tag{100}$$

Substituting the expression of $b_{T-1}(\beta,i_0)$ in Lemma 3.9 into Eq. (100), we can get the desired result (35).

Next, substituting Eq. (99) into Eqs. (33) and (34), according to $a_0(i_0)+\eta_0(i_0)=1$ and $\theta_0(i_0)=\eta_0(i_0)$, we obtain

$$\mathbb{E}_{0,x_0,i_0}\left(X_T^{\hat{\pi}^P}\right)=A_{0,T-1}\theta_0(i_0)x_0+b_{T-1}(\beta,i_0)+\frac{a_0(i_0)}{2\omega\eta_0(i_0)}+A_{0,T-1}a_0(i_0)x_0+b_{T-1}(\beta,i_0)\frac{a_0(i_0)}{\eta_0(i_0)}$$
$$=A_{0,T-1}x_0+\frac{a_0(i_0)}{2\omega\eta_0(i_0)}+\frac{b_{T-1}(\beta,i_0)}{\eta_0(i_0)}, \tag{101}$$

$$\mathbb{E}_{0,x_0,i_0}\left(\left(X_T^{\hat{\pi}^P}\right)^2\right)=A_{0,T-1}^2x_0^2+2\zeta(\beta)\eta_0(i_0)x_0+\frac{2x_0A_{0,T-1}b_{T-1}(\beta,i_0)a_0(i_0)}{\eta_0(i_0)}+\frac{x_0A_{0,T-1}a_0(i_0)}{\omega\eta_0(i_0)}+\frac{a_0(i_0)}{4\omega^2\eta_0^2(i_0)}+\frac{b_{T-1}^2(\beta,i_0)a_0(i_0)}{\eta_0^2(i_0)}+\frac{b_{T-1}(\beta,i_0)a_0(i_0)}{\omega\eta_0^2(i_0)}+\psi(\beta,i_0). \tag{102}$$

From Lemma 3.9 and $\theta_0(i_0)=\eta_0(i_0)$, we have

$$\zeta(\beta)\eta_0(i_0)-A_{0,T-1}b_{T-1}(\beta,i_0)=\eta_0(i_0)A_{0,T-1}\sum_{l=0}^{T-1}\chi_l(\beta)-A_{0,T-1}\theta_0(i_0)\sum_{m=0}^{T-1}\chi_m(\beta)=0. \tag{103}$$

Then from Eqs. (101)–(103), the variance of wealth at the terminal time T is

$$\mathrm{Var}_{0,x_0,i_0}\left(X_T^{\hat{\pi}^P}\right)=\mathbb{E}_{0,x_0,i_0}\left(\left(X_T^{\hat{\pi}^P}\right)^2\right)-\left[\mathbb{E}_{0,x_0,i_0}\left(X_T^{\hat{\pi}^P}\right)\right]^2$$
$$=2[\zeta(\beta)\eta_0(i_0)-A_{0,T-1}b_{T-1}(\beta,i_0)]x_0+\frac{a_0(i_0)}{4\omega^2\eta_0(i_0)}-\frac{b_{T-1}^2(\beta,i_0)}{\eta_0(i_0)}+\psi(\beta,i_0)$$
$$=\frac{a_0(i_0)}{4\omega^2\eta_0(i_0)}-\frac{b_{T-1}^2(\beta,i_0)}{\eta_0(i_0)}+\psi(\beta,i_0). \tag{104}$$

Eq. (101) together with the fact that $0<a_0(i_0)<1$ yields

$$\frac{1}{2\omega}=\frac{\left(\mathbb{E}_{0,x_0,i_0}\left(X_T^{\hat{\pi}^P}\right)-A_{0,T-1}x_0\right)\eta_0(i_0)-b_{T-1}(\beta,i_0)}{a_0(i_0)}. \tag{105}$$

Substituting Eq. (105) into Eq. (104), we have

$$\mathrm{Var}_{0,x_0,i_0}\left(X_T^{\hat{\pi}^P}\right)=\frac{\eta_0(i_0)}{a_0(i_0)}\left[\mathbb{E}_{0,x_0,i_0}\left(X_T^{\hat{\pi}^P}\right)-A_{0,T-1}x_0-\frac{b_{T-1}(\beta,i_0)}{\eta_0(i_0)}\right]^2-\frac{b_{T-1}^2(\beta,i_0)}{\eta_0(i_0)}+\psi(\beta,i_0)$$
$$=\frac{\eta_0(i_0)}{a_0(i_0)}\left[\mathbb{E}_{0,x_0,i_0}\left(X_T^{\hat{\pi}^P}\right)-A_{0,T-1}x_0-\sum_{m=0}^{T-1}\chi_m(\beta)\right]^2-\eta_0(i_0)\left(\sum_{m=0}^{T-1}\chi_m(\beta)\right)^2+\psi(\beta,i_0). \tag{106}$$

This is the conclusion (36). □

Appendix I. The proof of Lemma 4.2

Proof. First, we prove Eq. (46) by mathematical induction. For $k=T$, Eq. (46) holds obviously. Suppose that Eq. (46) holds for $T,T-1,\ldots,k+1$. Then,

$$\boldsymbol{\varpi}_{\mathbf{k}}=\mathbf{z}_{\mathbf{k}}+Q(k)\boldsymbol{\varpi}_{\mathbf{k+1}}=\mathbf{z}_{\mathbf{k}}+Q(k)\left(\sum_{m=k+1}^{T-1}\left(\prod_{j=k+1}^{m-1}Q(j)\right)\mathbf{z}_{\mathbf{m}}\right)=\sum_{m=k}^{T-1}\left(\prod_{j=k}^{m-1}Q(j)\right)\mathbf{z}_{\mathbf{m}},$$

which shows that Eq. (46) holds for k. Hence, Eq. (46) holds for all $k = 0, 1, \ldots, T$.

Next, we prove Eq. (47) also by mathematical induction. For $k = T$, Eq. (47) holds obviously. Suppose that Eq. (47) holds for $T, T-1, \ldots, k+1$. Then,

$$\begin{aligned}\mathbf{W_k} &= Q(k)\mathbf{W_{k+1}} + Q(k)\boldsymbol{\varpi}^2_{\mathbf{k+1}} - (Q(k)\boldsymbol{\varpi}_{\mathbf{k+1}})^2 \\ &= Q(k)\left(\sum_{m=k+2}^{T-1}\left(\prod_{j=k+1}^{m-1}Q(j)\right)\boldsymbol{\varpi}^2_{\mathbf{m}}\right. \\ &\quad \left. - \sum_{m=k+2}^{T-1}\left(\prod_{j=k+1}^{m-2}Q(j)\right)(Q(m-1)\boldsymbol{\varpi}_{\mathbf{m}})^2\right) \\ &\quad + Q(k)\boldsymbol{\varpi}^2_{\mathbf{k+1}} - (Q(k)\boldsymbol{\varpi}_{\mathbf{k+1}})^2 \\ &= \sum_{m=k+1}^{T-1}\left(\prod_{j=k}^{m-1}Q(j)\right)\boldsymbol{\varpi}^2_{\mathbf{m}} \\ &\quad - \sum_{m=k+1}^{T-1}\left(\prod_{j=k}^{m-2}Q(j)\right)(Q(m-1)\boldsymbol{\varpi}_{\mathbf{m}})^2, \end{aligned} \tag{107}$$

which means that Eq. (47) holds for k. So, Eq. (47) holds for all $k = 0, 1, \ldots, T-1$. □

Appendix J. The proof of Theorem 4.4

Proof. We prove this theorem by mathematical induction. For $k = T-1$, by Eqs. (1), (39) and (40), we have

$$\begin{aligned}&V_{T-1}(x_{T-1}, i) \\ &= \max_{\pi_{T-1}(i)}\Big\{\mathbb{E}_{T-1,x_{T-1},i}\left(V_T\left(X^\pi_T, \xi_T\right)\right) - \omega\mathbb{E}_{T-1,x_{T-1},i}\left(g^2_T\left(X^\pi_T, \xi_T\right)\right) \\ &\quad + \omega\big[\mathbb{E}_{T-1,x_{T-1},i}\left(g_T\left(X^\pi_T, \xi_T\right)\right)\big]^2\Big\} \\ &= \max_{\pi_{T-1}(i)}\Big\{\mathbb{E}_{T-1,x_{T-1},i}\left(X^\pi_T\right) - \omega\mathbb{E}_{T-1,x_{T-1},i}\left(\left(X^\pi_T\right)^2\right) \\ &\quad + \omega\big[\mathbb{E}_{T-1,x_{T-1},i}\left(X^\pi_T\right)\big]^2\Big\} \\ &= \max_{\pi_{T-1}(i)}\Bigg\{\mathbb{E}\left[A_{T-1,T-1}x_{T-1} + B_{T-1}(\beta) + \frac{S'_{T-1}(i)\pi_{T-1}(i)}{p_{T-1+y}}\right] \\ &\quad - \omega\mathbb{E}\left[\left(A_{T-1,T-1}x_{T-1} + B_{T-1}(\beta) + \frac{S'_{T-1}(i)\pi_{T-1}(i)}{p_{T-1+y}}\right)^2\right] \\ &\quad + \omega\left[\mathbb{E}\left(A_{T-1,T-1}x_{T-1} + B_{T-1}(\beta) + \frac{S'_{T-1}(i)\pi_{T-1}(i)}{p_{T-1+y}}\right)\right]^2\Bigg\} \\ &= \max_{\pi_{T-1}(i)}\Bigg\{A_{T-1,T-1}x_{T-1} + B_{T-1}(\beta) + \frac{s'_{T-1}(i)\pi_{T-1}(i)}{p_{T-1+y}} \\ &\quad - \omega\frac{\pi'_{T-1}(i)\left[\mathbb{E}\left(S_{T-1}(i)S'_{T-1}(i)\right) - s_{T-1}(i)s'_{T-1}(i)\right]\pi_{T-1}(i)}{p^2_{T-1+y}}\Bigg\} \\ &= A_{T-1,T-1}x_{T-1} + B_{T-1}(\beta) \\ &\quad + \max_{\pi_{T-1}(i)}\left\{\frac{s'_{T-1}(i)\pi_{T-1}(i)}{p_{T-1+y}} - \omega\frac{\pi'_{T-1}(i)\mathbf{cov}_{T-1}(i)\pi_{T-1}(i)}{p^2_{T-1+y}}\right\}. \end{aligned} \tag{108}$$

Since $\omega > 0$ and $\mathbf{cov}_{T-1}(i)$ is positive definite by Assumption 2.2, the application of the first order condition to $\pi_{T-1}(i)$ yields the following optimal solution

$$\hat{\pi}^E_{T-1}(i) = \frac{p_{T-1+y}}{2\omega}\mathbf{cov}^{-1}_{T-1}(i)s_{T-1}(i). \tag{109}$$

Substituting Eq. (109) into Eqs. (40) and (108) respectively, we obtain

$$\begin{aligned}V_{T-1}(x_{T-1}, i) &= A_{T-1,T-1}x_{T-1} + B_{T-1}(\beta) + \frac{s'_{T-1}(i)\mathbf{cov}^{-1}_{T-1}(i)s_{T-1}(i)}{4\omega} \\ &= A_{T-1,T-1}x_{T-1} + B_{T-1}(\beta) + \frac{1}{4\omega}z_{T-1}(i), \end{aligned} \tag{110}$$

and

$$\begin{aligned}g_{T-1}(x_{T-1}, i) &= \mathbb{E}_{T-1,x_{T-1},i}\left(g_T\left(X^\pi_T, \xi_T\right)\right) = \mathbb{E}_{T-1,x_{T-1},i}\left(X^\pi_T\right) \\ &= A_{T-1,T-1}x_{T-1} + B_{T-1}(\beta) + \frac{1}{2\omega}z_{T-1}(i). \end{aligned} \tag{111}$$

Because $\chi_{T-1}(\beta) = B_{T-1}(\beta)$, $\varpi_{T-1}(i) = z_{T-1}(i)$ and $W_{T-1}(i) = 0$, Eqs. (109)–(111) show that Eqs. (48)–(50) hold for $k = T-1$.

Now suppose that Eqs. (48)–(50) hold for $T-1, T-2, \ldots, k+1$. Then, for k, by the extended Bellman equation (39), we have

$$\begin{aligned}&V_k(x_k, i) \\ &= \max_{\pi_k(i)}\Big\{\mathbb{E}_{k,x_k,i}\left(V_{k+1}\left(X^\pi_{k+1}, \xi_{k+1}\right)\right) - \omega\mathbb{E}_{k,x_k,i}\left(g^2_{k+1}\left(X^\pi_{k+1}, \xi_{k+1}\right)\right) \\ &\quad + \omega\big[\mathbb{E}_{k,x_k,i}\left(g_{k+1}\left(X^\pi_{k+1}, \xi_{k+1}\right)\right)\big]^2\Big\} \\ &= \max_{\pi_k(i)}\Bigg\{\mathbb{E}_{k,x_k,i}\Bigg[A_{k+1,T-1}X^\pi_{k+1} + \sum_{l=k+1}^{T-1}\chi_l(\beta) \\ &\quad + \frac{1}{4\omega}\left(\varpi_{k+1}(\xi_{k+1}) - W_{k+1}(\xi_{k+1})\right)\Bigg] \\ &\quad - \omega\mathbb{E}_{k,x_k,i}\left[\left(A_{k+1,T-1}X^\pi_{k+1} + \sum_{l=k+1}^{T-1}\chi_l(\beta) + \frac{1}{2\omega}\varpi_{k+1}(\xi_{k+1})\right)^2\right] \\ &\quad + \omega\left[\mathbb{E}_{k,x_k,i}\left(A_{k+1,T-1}X^\pi_{k+1} + \sum_{l=k+1}^{T-1}\chi_l(\beta) + \frac{1}{2\omega}\varpi_{k+1}(\xi_{k+1})\right)\right]^2\Bigg\} \\ &= \max_{\pi_k(i)}\Bigg\{A_{k,T-1}x_k + \sum_{l=k}^{T-1}\chi_l(\beta) \\ &\quad + \frac{1}{4\omega}\mathbb{E}_{k,x_k,i}\left[\varpi_{k+1}(\xi_{k+1}) - W_{k+1}(\xi_{k+1})\right] \\ &\quad - \omega\frac{A^2_{k+1,T-1}\pi'_k(i)\left[\mathbb{E}\left(S_k(i)S'_k(i)\right) - s_k(i)s'_k(i)\right]\pi_k(i)}{p^2_{k+y}} \\ &\quad + \frac{A_{k+1,T-1}s'_k(i)\pi_k(i)}{p_{k+y}} \\ &\quad - \frac{1}{4\omega}\left[\mathbb{E}_{k,x_k,i}\left(\varpi^2_{k+1}(\xi_{k+1})\right) - \mathbb{E}^2_{k,x_k,i}\left(\varpi_{k+1}(\xi_{k+1})\right)\right] \\ &\quad - \frac{A_{k+1,T-1}\left[\mathbb{E}_{k,x_k,i}\left(\varpi_{k+1}(\xi_{k+1})S'_k(i)\right) - \mathbb{E}_{k,x_k,i}\left(\varpi_{k+1}(\xi_{k+1})\right)s'_k(i)\right]\pi_k(i)}{p_{k+y}}\Bigg\} \\ &= \max_{\pi_k(i)}\Bigg\{A_{k,T-1}x_k + \sum_{l=k}^{T-1}\chi_l(\beta) + \frac{A_{k+1,T-1}s'_k(i)\pi_k(i)}{p_{k+y}} \\ &\quad + \frac{1}{4\omega}\sum_{j=1}^{J}q_{ij}(k)\left[\varpi_{k+1}(j) - W_{k+1}(j)\right] \\ &\quad - \omega\frac{A^2_{k+1,T-1}\pi'_k(i)\mathbf{cov}_k(i)\pi_k(i)}{p^2_{k+y}} \\ &\quad - \frac{1}{4\omega}\left(\sum_{j=1}^{J}q_{ij}(k)\varpi^2_{k+1}(j) - \left(\sum_{j=1}^{J}q_{ij}(k)\varpi_{k+1}(j)\right)^2\right)\Bigg\}, \end{aligned} \tag{112}$$

where the last equality uses the conclusion $\mathbb{E}_{k,x_k,i}\left(\varpi_{k+1}(\xi_{k+1})S'_k(i)\right) - \mathbb{E}_{k,x_k,i}\left(\varpi_{k+1}(\xi_{k+1})\right)s'_k(i) = 0$, which holds due to the fact that $\varpi_{k+1}(\xi_{k+1})$ and $S'_k(i)$ are statistically independent. Because $\omega > 0$, $A^2_{k+1,T-1} > 0$ and $\mathbf{cov}_k(i)$ is positive definite by Assumption 2.2, the application of the first order condition to $\pi_k(i)$ yields the optimal solution

$$\hat{\pi}_k^E(i) = \frac{p_{k+y}}{2\omega A_{k+1,T-1}} \mathbf{cov}_k^{-1}(i) s_k(i). \tag{113}$$

Substituting Eq. (113) into Eqs. (112) and (40) respectively, we obtain

$$\begin{aligned} & V_k(x_k, i) \\ &= A_{k,T-1}x_k + \sum_{l=k}^{T-1} \chi_l(\beta) + \frac{1}{4\omega}\sum_{j=1}^{J} q_{ij}(k)\left[\varpi_{k+1}(j) - W_{k+1}(j)\right] \\ &\quad + \frac{s_k'(i)\mathbf{cov}_k^{-1}(i)s_k(i)}{4\omega} \\ &\quad - \frac{1}{4\omega}\left(\sum_{j=1}^{J} q_{ij}(k)\varpi_{k+1}^2(j) - \left(\sum_{j=1}^{J} q_{ij}(k)\varpi_{k+1}(j)\right)^2\right) \\ &= A_{k,T-1}x_k + \sum_{l=k}^{T-1} \chi_l(\beta) + \frac{1}{4\omega}\left(\varpi_k(i) - W_k(i)\right), \end{aligned} \tag{114}$$

and

$$\begin{aligned} g_k(x_k, i) &= \mathbb{E}_{k,x_k,i}\left(g_{k+1}\left(X_{k+1}^{\pi}, \xi_{k+1}\right)\right) \\ &= \mathbb{E}_{k,x_k,i}\left(A_{k+1,T-1}X_{k+1}^{\pi} + \sum_{l=k+1}^{T-1}\chi_l(\beta) + \frac{1}{2\omega}\varpi_{k+1}(\xi_{k+1})\right) \\ &= A_{k,T-1}x_k + \sum_{l=k}^{T-1}\chi_l(\beta) + \frac{1}{2\omega}z_k(i) + \frac{1}{2\omega}\sum_{j=1}^{J} q_{ij}(k)\varpi_{k+1}(j) \\ &= A_{k,T-1}x_k + \sum_{l=k}^{T-1}\chi_l(\beta) + \frac{1}{2\omega}\varpi_k(i). \end{aligned} \tag{115}$$

Eqs. (113)–(115) show that Eqs. (48)–(50) hold for k. Hence, Eqs. (48)–(50) hold for all $k = 0, 1, \ldots, T-1$. □

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