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Interfaces with Other Disciplines

# Non-zero-sum stochastic differential reinsurance and investment games with default risk

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ABSTRACT

This paper investigates the implications of strategic interaction (i.e., competition) between two CARA insurers on their reinsurance-investment policies. The two insurers are concerned about their terminal wealth and the relative performance measured by the difference in their terminal wealth. The problem of finding optimal policies for both insurers is modelled as a non-zero-sum stochastic differential game. The reinsurance premium is calculated using the variance premium principle and the insurers can invest in a risk-free asset, a risky asset with Heston's stochastic volatility and a defaultable corporate bond. We derive the Nash equilibrium reinsurance policy and investment policy explicitly for the game and prove the corresponding verification theorem. The equilibrium strategy indicates that the best response of each insurer to the competition is to mimic the strategy of its opponent. Consequently, either the reinsurance strategy or the investment strategy of an insurer with the relative performance concern is riskier than that without the concern. Numerical examples are provided to demonstrate the findings of this study.



## 1. Introduction

Studies of optimal reinsurance and/or investment decisions are becoming a significant portion of the mainstream research of insurance and actuarial science. In a regular framework, an insurer is assumed to purchase reinsurance contracts from a reinsurer to reduce the risk of random individual claims, while the insurer may invest in a financial market for a higher rate of return or to hedge the risk of the claims, under certain optimality rules. Among many others, for example, we refer to Browne (1995) and Azcue and Muler (2013) with respect to minimizing the ruin probability, Yang and Zhang (2005) and Zhu, Deng, Yue, and Deng (2015) with respect to maximizing the utility of terminal wealth, and Chiu and Wong (2012) and Bi, Meng, and Zhang (2014) with respect to the mean-variance criterion.

The majority of these studies do not consider the strategic interaction (i.e., competition) among insurers. In this study, we consider two competitive insurance companies that are concerned with their relative wealth.[1] The two companies competitively select their reinsurance and investment policies to maximize their utilities based on their terminal wealth and relative wealth. We analyze their optimal reinsurance and investment policies within a non-zero-sum differential game framework.

This paper is related with prior studies of stochastic differential games. Elliott (1976) analyzed the relationship between the existence of the value of a zero-sum stochastic differential game and the Isaacs condition. Elliott's study was followed by many others including Zhang and Siu (2009) and Elliott and Siu (2011) about zero-sum stochastic differential games between the investor/insurer and market. Browne (2000), Zeng (2010), and Taksar and Zeng (2011) considered zero-sum stochastic differential games between two competitive investors/insurers. By contrast, Bensoussan and Frehse (2000) examined the non-zero-sum stochastic differential game with $N$ players over an infinite time horizon. By applying dynamic programming techniques, the Nash equilibrium can be constructed as a solution to a system of parabolic partial differential equations. Espinosa and Touzi (2015) developed a non-zero-sum stochastic investment game with

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[1] Regarding more discussions on relative performance concern and its implications, we refer to Corneo and Olivier (1997), DeMarzo, Kaniel, and Kremer (2008), and Basak and Makarov (2014).

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$N$ players who consider their relative performances against their peers. Their study showed the existence and uniqueness of the Nash equilibrium for the cases of unconstrained and constrained agents with exponential utilities within a Black-Scholes market framework. Bensoussan, Siu, Yam, and Yang (2014) formulated a non-zero-sum stochastic differential investment and reinsurance game between two insurance companies whose surplus processes were modulated by continuous-time Markov chains. More studies on non-zero-sum invest/reinsurance games may be found in Dang and Forsyth (2016), Villena and Reus (2016), and Pun, Siu, and Wong (2016).

The present study is motivated by Bensoussan et al. (2014) and Espinosa and Touzi (2015), who investigated Black-Scholes financial markets. In this study, we consider a more general investment opportunity set that contains a risky asset (i.e., stock) and a corporate bond that is defaultable. The majority of the aforementioned papers assumed Black-Scholes financial markets with constant volatilities; but it is well-known that such an assumption is unrealistic. We thus relax this assumption and assume that the stock price follows a Heston stochastic volatility model. The Heston model can explain a number of important empirical features of real market data such as volatility clustering, fat tails of return distributions and "volatility smile" (see Heston, 1993). Employing the Heston model, Li, Zeng, and Lai (2012) derived an optimal time-consistent investment and a proportional reinsurance policy under a mean-variance criterion within a game framework. Zhao, Rong, and Zhao (2013) obtained a closed-form expression for the optimal excess-of-loss reinsurance and investment policy when the surplus could be characterized as a jump-diffusion process.

In addition to the stochastic volatility stock, we assume that insurers can invest into a defaultable bond. As a consequence of the financial crisis in 2008, investors and regulators have been paying more attention to default risk management, whereas the corporate bond market continues to develop. "The global corporate bond markets have almost tripled in size since 2000, reaching 49 trillion in 2013."[2]

There are two approaches in the existing literature that are used to model default. The first is the so-called structural approach. In this approach, a corporate bond is regarded as a contingent claim on the value of a firm, and the default event occurs as the first hitting time of the firm value on a given barrier. For more information about this approach, we refer to Merton (1974), Korn and Kraft (2003) and Lakner and Liang (2008). The second is the so-called reduced-form approach. The default time $\tau$ is modelled as the first jump of a Poisson point process. Many prior studies solved the problem of optimal portfolio/consumption choice using the reduced-form model (e.g., Bielecki and Jang, 2006, Bo, Wang, and Yang, 2013 , Capponi and Figueroa López, 2014 and Sun, Aw, Loxton, & Teo, 2017). Our study adopts the reduced-form model because it is more flexible.

The contributions of this paper are summarized as follows. We derive the equilibrium strategies of the game with default risk and obtain the corresponding value functions via the dynamic programming approach. We find that when competition is considered, each insurance company will adopt a strategy riskier than that when no competition is involved. In addition, it is optimal to invest a positive amount in the defaultable market if the asset's risk premium is positive. The sensitivities of the equilibrium strategies regarding model parameters are investigated and a verification theorem is provided.

[2] From *Corporate Bond Markets: a Global Perspective* by Tendulkar and Hancock (2014), Staff Working Paper of the IOSCO Research Department. Available at http://www.iosco.org/research/pdf/swp/SW4-Corporate-Bond-Markets.

Our paper differs from Bensoussan et al. (2014) in at least four respects. First, we extend the models of Bensoussan et al. (2014) by considering a more general financial market. In particular, the financial market in our study is assumed to contain a risky asset with a Heston stochastic volatility and a defaultable corporate bond rather than the standard Black-Scholes market used by Bensoussan et al. (2014). As a result, our study discloses effects of defaultable bonds and stochastic volatility on investment. We demonstrate that insurers will reduce investment in the risk-free bond in the presence of a defaultable bond and modify investment strategies of the risky asset against unfavourable changes of the stochastic volatility process.

Second, in accordance with the expected value criterion, if the expectations of two claims are the same, the same premium will be charged. However, the two claims may have different volatilities hence contain different risk. Compared with the expected value criterion, the variance premium principle considering the volatility of random individual loss may be more realistic than the expected premium criteria. The reinsurance premium in our study is calculated using the variance premium principle rather than the expected premium principle in Bensoussan et al. (2014).

Third, we obtain the equilibrium strategies in a more general environment as well as a detailed analysis on the impacts of competition on insurers' reinsurance-investment rules. We find the herd effect on insurers' decisions, that is, each insurers will make decisions by mimicking its opponent's strategy. In contrast, Bensoussan et al. (2014) focus on deriving the equilibrium strategies for the non-zero-sum game, not on a comprehensive analysis of the impacts of competition on insurers' decision-making behaviours.

Finally, to prove verification theorems, Bensoussan et al. (2014) make a uniform Lipschitz condition assumption on the coefficients of the asset dynamic processes. Because the Heston model does not satisfy this uniform Lipschitz condition, we provide a verification theorem with a locally Lipschitz condition using Zeng and Taksar's (2013) approach.

The remainder of this paper is organized as follows. Section 2 introduces the basic model setup of each insurance company. Section 3 formulates the non-zero-sum stochastic differential reinsurance and investment optimization game between two competitive insurance companies. In Section 4, we derive the HJB equation for the pre- and post-default cases; then, the explicit expressions for Nash equilibrium strategies and the corresponding value functions are obtained. In addition, we prove the corresponding verification theorem. In Section 5, numerical examples are provided and finally, we conclude the study in Section 6.

## 2. Model setup

### 2.1. Dynamics of the financial assets

We consider a financial market that consists of a risk-free bond, a risky asset (i.e., a stock) and a corporate zero coupon bond that is defaultable. The price processes are denoted by $\{S_0(t)\}_{t\geq 0}$, $\{S(t)\}_{t\geq 0}$ and $\{p(t, T_1)\}_{t\geq 0}$, where $T_1$ is a fixed time horizon. The processes are defined in a completely filtered probability space $(\Omega, \mathcal{G}, \mathbb{P})$, where $\mathbb{P}$ is the real-world probability measure, $\mathcal{G} := \{\mathcal{G}_t\}_{t\geq 0}$ is an enlarged filtration given by $\mathcal{G}_t = \mathcal{F}_t \vee \sigma\{H_s : 0 \leq s \leq t\}$(the filtrations $\mathcal{F}_t$ and $\mathcal{H}_t$ will be introduced later). Let $B_1(t)$, $B_2(t)$, $W_1(t)$ and $W_2(t)$ be four standard Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$. The natural filtration $\mathcal{F} := \{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual hypotheses of completeness and right continuity.

The price process of the risk-free asset is given by

$$S_0(t) = rS_0(t)dt, \tag{2.1}$$

where $r$ is a constant. The stock price follows a Heston stochastic volatility model (see Section 2 in Liu, 2007):

$$\begin{cases} dS(t) = S(t)\Big[(r+\alpha L(t))dt + \sqrt{L(t)}dW_1(t)\Big], \\ dL(t) = \mathcal{K}(\beta - L(t))dt + \nu\sqrt{L(t)}dW_2(t), \end{cases} \quad (2.2)$$

where $W_1(t)$ and $W_2(t)$ are standard Brownian motions; $E[dW_1(t)dW_2(t)] = \hat{\rho}dt$, $S(0) = s > 0; L(0) = l > 0$; $\beta > 0$ is the long run average of the variance process; and $\nu > 0$ is the volatility of the variance. We need $2\mathcal{K}\beta \geq \nu^2$ to ensure that $L(t)$ is non-negative almost surely.

Next, we model the price process for a corporate zero coupon bond directly under the real-world probability measure $\mathbb{P}$, following Bielecki and Jang (2006).

We assume that $T_1$ is the maturity date of the corporate bond. This bond is defaultable, we assume that the market value is recovered at the default time, as done by Duffie and Singleton (1999). Let $\tau$ denotes the default time of the corporate bond. We assume that $\tau$ is the first jump time of a Poisson process with constant jump intensity $h^{\mathbb{Q}}$ under a risk-neutral measure $\mathbb{Q}$, which is equivalent to the real-world probability measure $\mathbb{P}$. The default process is $H(t) = \mathbf{I}_{\{\tau \leq t\}}$. Let $\mathcal{G}_t$ be the smallest filtration containing the reference filtration $\mathcal{F}_t$ and under which $\tau$ is a stopping time; this means that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t = \mathcal{F}_t \vee \sigma\{H_s : 0 \leq s \leq t\}$. Such an information structure is standard in the reduced-form approach. We formulate this model under the martingale invariance property, which is generally called the (**H**) hypothesis (see Section 6.1.1 in Bielecki and Rutkowski, 2001, and Proposition 1 in Blanchet-Scalliet & Jeanblanc, 2004): under the real-world probability measure $\mathbb{P}$, every square-integrable $\mathcal{F}_t$-martingale is also a square-integrable martingale under the enlarged filtration $\mathcal{G}_t$.

In the case of default, the investor recovers a fraction of the market value of the defaultable bond just prior to default. Let $\zeta \in (0, 1)$ denote the constant loss rate of the corporate bond. In alignment with Lemma 3 in Bielecki and Jang (2006), we assume the corporate bond price is given by

$$\begin{aligned} dp(t, T_1) = {} & p(t-, T_1)[rdt + (1 - H(t-))\eta(1 - \Delta)dt \\ & -(1 - H(t-))\zeta dM^P(t)], \end{aligned} \quad (2.3)$$

where we use

$$\begin{cases} p(t, T_1) = e^{-(r+\eta)(T_1 - t)} & \text{if } t \in [0, \tau \wedge T_1], \\ p(t, T_1) = (1-\zeta)e^{-(r+\eta)(T_1-\tau)}e^{r(t-\tau)} & \text{if } t \in [\tau \wedge T_1, T_1], \end{cases} \quad (2.4)$$

$M^P(t) = H(t) - h^{\mathbb{Q}}\int_0^t \Delta(1 - H(u-))du$ is a $\mathcal{G}$-martingale under $\mathbb{P}$; $\eta = h^{\mathbb{Q}}\zeta$ is the credit spread under the real-world probability measure; $\frac{1}{\Delta} \geq 1$ denotes the constant default risk premium and the arrival intensity of the default under the measure $\mathbb{Q}$ is given by $h^{\mathbb{Q}} = h^P/\Delta$ .

### 2.2. Dynamics of the surplus processes

Suppose that there are two competing insurance companies, whose reserve processes $U_k(t)$, $k = 1, 2$ are modelled by

$$U_k(t) = u_k + d_k t - C_k(t), \quad for\ k = 1, 2,$$
$$U_k(0) = u_k \geq 0,$$

where $d_k$ is the constant rate of the premium received by the insurance company $k$ and the aggregate claims processes for the two companies are given by

$$C_1(t) = \sum_{i=1}^{N_1(t)+N(t)} X_i \quad and \quad C_2(t) = \sum_{i=1}^{N_2(t)+N(t)} Y_i,$$

where $C_1(t)$ and $C_2(t)$ are two compound Poisson processes defined on $(\Omega, \mathcal{F}, \mathbb{P})$. $N_1(t)$, $N_2(t)$ and $N(t)$ are three mutually independent and homogeneous Poisson processes with intensities $\lambda_1$, $\lambda_2$, and $\lambda$, respectively. The claim's sizes $\{X_i\}_{i\in\mathbb{N}^+}$ ($\{Y_i\}_{i\in\mathbb{N}^+}$) are independent of $N(t)$, $N_1(t)$ and $N_2(t)$; they are i.i.d. random variables with a continuous distribution function $F_X$ ($F_Y$) and finite first and second moments $\mu_1(\mu_2)$ and $\sigma_1^0(\sigma_2^0)$.

We now consider the situation where each insurance company can transfer the claim risk by continuously purchasing proportional reinsurance. Then, the corresponding reserve process for insurer $k$ becomes:

$$dU_k^{q_k}(t) = (d_k - \delta(q_k(t)))dt - q_k(t)dC_k(t), \quad for\ k = 1, 2, \quad (2.5)$$

where $q_k(t) \in [0, 1]$ represents the proportion insured by insurer $k$; thus, $1 - q_k(t)$ is the proportion reinsured to the reinsurance company. Assume that the constant reinsurance premium $\delta(q_k(t))$ is calculated by the variance principle; thus, insurer $k$ should pay a reinsurance premium at a continuous rate $\delta(q_k(t)) = (1 - q_k(t))n_k + \Lambda(1 - q_k(t))^2\sigma_k^2$ with a safety loading of $\Lambda > 0$, where $n_1 = (\lambda_1 + \lambda)E(X_i)$, $n_2 = (\lambda_2 + \lambda)E(Y_i)$, $\sigma_1^2 = (\lambda_1 + \lambda)E(X_i^2)$ and $\sigma_2^2 = (\lambda_2 + \lambda)E(Y_i^2)$. As in Section 3 of Grandell (1977), the compound Poisson process $C_i$ can be approximated by Brownian motion with drift:

$$C_k(t) \approx n_k t - \sigma_k B_k(t),$$

where $B_1(t)$ and $B_2(t)$ are standard Brownian motions with the correlation coefficient

$$\rho = \frac{\lambda E(X_i)E(Y_i)}{\sqrt{(\lambda_1 + \lambda)E(X_i^2)(\lambda_2 + \lambda)E(Y_i^2)}} = \frac{\lambda\mu_1\mu_2}{\sigma_1\sigma_2}.$$

Therefore, $E[B_1(t)B_2(t)] = \rho t$ and the continuous-time dynamics of the reserve process for the insurer $k$ is finally formulated as

$$d\hat{U}_k^{q_k}(t) = (d_k - \delta(q_k(t)) - q_k(t)n_k)dt + q_k(t)\sigma_k dB_k(t), k = 1, 2.$$

### 2.3. Wealth processes

In this model, we assume that each insurer continuously invests in the risk-free bond, the stock and the corporate zero coupon bond, and purchases reinsurance contracts from the same reinsurance company. The investment horizon is $[0, T]$ and $T < T_1$. Thus, in the defaultable financial market described above, the wealth process of insurer $k$ is given by

$$\begin{aligned} dZ_k^{\pi_k}(t) = {} & \frac{(Z_k^{\pi_k}(t) - \theta_k(t) - \gamma_k(t))}{S_0(t)}dS_0(t) + \frac{\theta_k(t)}{S(t)}dS(t) \\ & + \frac{\gamma_k(t)}{p(t-, T_1)}dp(t, T_1) + (d_k - \delta(q_k(t)) \\ & - q_k(t)n_k)dt + \sigma_k q_k(t)dB_k(t) \\ = {} & [rZ_k^{\pi_k}(t) + (d_k - \delta(q_k(t)) - q_k(t)n_k) + \theta_k(t)\alpha L(t) \\ & + \gamma_k(t)(1 - H(t-))\eta(1 - \Delta)]dt + \theta_k(t)\sqrt{L(t)}dW_1(t) \\ & + \sigma_k q_k(t)dB_k(t) - \gamma_k(t)\zeta(1 - H(t-))dM^P(t), \end{aligned} \quad (2.6)$$

where $\theta_k(t)$ and $\gamma_k(t)$ represent the dollar amounts of insurer $k$'s wealth invested in the stock and the corporate bond, respectively; and $1 - q_k(t)$ denotes the proportional reinsurance purchased by insurer $k$ at time $t$. We assume that the corporate bond is not traded after default. Let $\pi_k(t) = (q_k(t), \theta_k(t), \gamma_k(t))$ be a reinsurance-investment strategy followed by insurer $k$.

**Definition 2.1.** A tripe process $\{\pi_k(t)\}_{t\in[0, T]}$ is an admissible strategy if

(1) $\pi_k(t)$ is a $\mathcal{G}_t$-progressively measurable process;
(2) $\int_0^T [\sigma_k^2 q_k(t)^2 + \theta_k(t)^2 L(t)]dt < \infty$;
(3) Under $\pi_k$, the SDE (2.6) has a unique strong solution.

Let $\Pi_k$ denotes the space of all admissible strategies.

## 3. Formulation of a non-zero-sum game

Both of the insurance companies choose an admissible reinsurance and investment strategy $\pi_k$ to maximize their own terminal wealth. Each insurer cares about the difference between its terminal wealth and the other's, and tries to perform better relative to its competitor. We formulate this optimization problem as a non-zero sum stochastic differential game between the two competitive insurers. We only consider games with perfect revelation, or perfect observation, such that the insurers' choices are instantaneously revealed to its opponent. The game (i.e., the competition) terminates at time $T$.

Given $Z_k^{\pi_k}(t)=z_k, Z_j^{\pi_j}(t)=z_j, L(t)=l, H(t)=h$, we define a non-zero-sum stochastic differential game with the following payoff (objective) functions (see Bensoussan et al., 2014 or Espinosa & Touzi, 2015) for $j\neq k\in\{1, 2\}$,

$$\begin{aligned} J_k^{(\pi_k,\pi_j)}&(t,z_k,z_j,l,h)\\ &=E\Big[\mathcal{U}_k\Big((1-\omega_k)Z_k^{\pi_k}(T)+\omega_k(Z_k^{\pi_k}(T)-Z_j^{\pi_j}(T))\Big)\Big|(Z_k^{\pi_k}(t),\\ &\quad Z_j^{\pi_j}(t),L(t),H(t))=(z_k,z_j,l,h)\Big]\\ &=E\Big[\mathcal{U}_k\Big(Z_k^{\pi_k}(T)-\omega_k Z_j^{\pi_j}(T)\Big)\Big|(Z_k^{\pi_k}(t),Z_j^{\pi_j}(t),L(t),H(t))\\ &=(z_k,z_j,l,h)\Big], \end{aligned} \tag{3.1}$$

where $\mathcal{U}_k$ is a strictly increasing and strictly concave smooth utility function for insurer (i.e., player) $k$ (i.e., $\mathcal{U}_k'>0$ and $\mathcal{U}_k''<0$). The parameter $\omega_k\in[0,1], k=1,2,$ describes insurer $k$'s performance relative to its competitor $j(j\neq k\in\{1, 2\})$. A greater $\omega_k$ means that insurer $k$ cares more about its relative wealth.

**Problem 3.1.** The classical non-zero-sum stochastic differential game problem is to find a Nash equilibrium $(\pi_1^*,\pi_2^*)\in\Pi_1\times\Pi_2$ such that

$$J_1^{(\pi_1^*,\pi_2^*)}(t,z_1,z_2,l,h)\geq J_1^{(\pi_1,\pi_2^*)}(t,z_1,z_2,l,h), \tag{3.2}$$

and

$$J_2^{(\pi_1^*,\pi_2^*)}(t,z_1,z_2,l,h)\geq J_2^{(\pi_1^*,\pi_2)}(t,z_1,z_2,l,h). \tag{3.3}$$

If (3.2) and (3.3) hold, then we respectively define the value functions of insurers 1 and 2 as follow:

$$J_1(t,z_1,z_2,l,h)=J_1^{(\pi_1^*,\pi_2^*)}(t,z_1,z_2,l,h)=\sup_{\pi_1\in\Pi_1}J_1^{(\pi_1,\pi_2^*)}(t,z_1,z_2,l,h) \tag{3.4}$$

and

$$J_2(t,z_1,z_2,l,h)=J_2^{(\pi_1^*,\pi_2^*)}(t,z_1,z_2,l,h)=\sup_{\pi_2\in\Pi_2}J_2^{(\pi_1^*,\pi_2)}(t,z_1,z_2,l,h). \tag{3.5}$$

We refer the admissible strategies $\pi_1^*$ and $\pi_2^*$ as the competitively optimal reinsurance and investment strategies.

To establish a Nash equilibrium for the above problem, we first define that $\widehat{Z}_k^{\pi_k}(t)\triangleq Z_k^{\pi_k}(t)-\omega_k Z_j^{\pi_j}(t)$ after fixing $\pi_j$ for $j\neq k\in\{1, 2\}$, and it readily follows that:

$$\begin{aligned} d\widehat{Z}_k^{\pi_k}(t)&=[r\widehat{Z}_k^{\pi_k}(t)+(d_k-\omega_k d_j)-(\delta(q_k(t))-\omega_k\delta(q_j(t)))\\ &\quad-(q_k(t)n_k-\omega_k q_j(t)n_j)+(\theta_k(t)-\omega_k\theta_j(t))\alpha L(t)\\ &\quad+(\gamma_k(t)-\omega_k\gamma_j(t))(1-H(t))\eta(1-\Delta)]dt\\ &\quad+(\theta_k(t)-\omega_k\theta_j(t))\sqrt{L(t)}dW_1(t)+\sigma_k q_k(t)dB_k(t)\\ &\quad-\omega_k\sigma_j q_j(t)dB_j(t)-(\gamma_k(t)\\ &\quad-\omega_k\gamma_j(t))\zeta(1-H(t-))dM^P(t), \end{aligned} \tag{3.6}$$

with $\widehat{Z}_k^{\pi_k}(0)=z_k-\omega_k z_j$.

For $L(t)=l, H(t)=h$ and $\widehat{Z}_k^{\pi_k}(t)=z_k-\omega_k z_j$, where $0\leq t\leq T$ and $k\neq j\in\{1, 2\}$, let:

$$\begin{aligned} J_k(t,\hat{z}_k,l,h)&\triangleq\sup_{\pi_k\in\Pi_k}E\Big[\mathcal{U}_k\Big(Z_k^{\pi_k}(T)-\omega_k Z_j^{\pi_j^*}(T)\Big)|\widehat{Z}_k^{\pi_k}(t)\\ &=\widehat{z}_k, L(t)=l, H(t)=h\Big], \end{aligned} \tag{3.7}$$

be the value function in $\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+\times\{0,1\}$.

## 4. Solution to the non-zero-sum game for CARA preference

Compared to an individual investor, the insurance company has considerable wealth, and thus, the risk aversion coefficient is relatively stable and can be regarded as a constant value. Moreover, the ruin event may occur for an insurer, as a consequence, the insurer's wealth may be negative. In view of this fact, we assume insurers with exponential utility preferences.

Suppose that insurer $k$ has the following exponential utility function

$$\mathcal{U}_k(\widehat{z}_k)=-\frac{1}{m_k}e^{-m_k\widehat{z}_k},\quad for\ k=1,2,$$

where $m_k$ is a positive constant(i.e., a constant absolute risk aversion coefficient). Using standard dynamic programming techniques, we see that the value function $J_k$ satisfies the following HJB partial differential equation:

$$\begin{cases}\sup_{\pi_k\in\Pi_k}\mathcal{L}_k^{\pi_k}J_k(t,\widehat{z}_k,l,h)=0,\\ J_k(T,\widehat{z}_k,l,h)=\mathcal{U}_k(\widehat{z}_k),\end{cases} \tag{4.1}$$

for all $t\in[0, T]$, where

$$\begin{aligned} &\mathcal{L}_k^{\pi_k}J_k(t,\widehat{z}_k,l,h)\\ &=\frac{\partial J_k(t,\widehat{z}_k,l,h)}{\partial t}+\Big\{\Big[r\widehat{z}_k+(d_k-\omega_k d_j)-(\delta(q_k(t))-\omega_k\delta(q_j^*(t)))\\ &\quad-(q_k(t)n_k-\omega_k q_j^*(t)n_j)+(\theta_k(t)-\omega_k\theta_j^*(t))\alpha l\\ &\quad+(\gamma_k(t)-\omega_k\gamma_j^*(t))\times(1-h)\eta\Big]\frac{\partial J_k(t,\widehat{z}_k,l,h)}{\partial\widehat{z}_k}\\ &\quad+\frac{1}{2}\Big[(\theta_k(t)-\omega_k\theta_j^*(t))^2 l+\sigma_k^2 q_k^2(t)+\omega_k^2\sigma_j^2 q_j^{*2}(t)\\ &\quad-2\rho\omega_k\sigma_k\sigma_j q_k(t)q_j^*(t)\Big]\frac{\partial^2 J_k(t,\widehat{z}_k,l,h)}{\partial\widehat{z}_k^2}+\mathcal{K}(\beta-l)\frac{\partial J_k(t,\widehat{z}_k,l,h)}{\partial l}\\ &\quad+\frac{1}{2}\nu^2 l\frac{\partial^2 J_k(t,\widehat{z}_k,l,h)}{\partial l^2}+\hat{\rho}(\theta_k(t)-\omega_k\theta_j^*(t))\nu l\frac{\partial^2 J_k(t,\widehat{z}_k,l,h)}{\partial\widehat{z}_k\partial l}\\ &\quad+\Big(J_k(t,\widehat{z}_k-(\gamma_k(t)-\omega_k\gamma_j^*(t))\zeta,l,h+1)-J_k(t,\widehat{z}_k,l,h)\Big)h^P(1-h)\Big\}. \end{aligned}$$

We solve this nonlinear HJB equation by two steps.

Step 1. We split the original value function into two pieces that represent the pre- and post-default value functions:

$$J_k(t,\widehat{z}_k,l,h)=\begin{cases}J_k(t,\widehat{z}_k,l,0), & \text{if } h=0\quad\text{(the pre-default case)},\\ J_k(t,\widehat{z}_k,l,1), & \text{if } h=1\quad\text{(the post-default case)}.\end{cases}$$

Step 2. We reduce the HJB Eq. (4.1) into two simple HJB equations that are satisfied by the pre-default value function and the post-default value function, respectively. Then, we solve the HJB equation for the post-default value function using the standard dynamic programming approach. After that, the pre-default value function is solved.

### 4.1. Equilibrium strategy after default

In this subsection, we derive the Nash equilibrium reinsurance-investment strategy to characterize the insurers' strategic interactions with the relative performance concerns after default.

Theorem 4.1 describes the post-default equilibrium strategy and its associated value functions.

**Theorem 4.1.** *[Post-default] For any $t \in [\tau \wedge T, T]$, the equilibrium investment strategy of stock asset is described by*

$$\theta_k^*(t) = \frac{\widehat{\theta}_k^*(t) + \omega_k \widehat{\theta}_j^*(t)}{1 - \omega_k \omega_j}, \quad \text{for } k \neq j \in \{1, 2\}, \tag{4.2}$$

*where $\widehat{\theta}_k^*(t) = (\frac{\alpha}{m_k} + \frac{f_k(t)\widehat{\rho}\nu}{m_k})e^{-r(T-t)}$. The equilibrium investment strategy of corporate bond is*

$$\gamma_k^*(t) = 0, \quad \text{for } k = 1, 2. \tag{4.3}$$

*For the equilibrium reinsurance policy $q_k^*(t), k = 1, 2$, we define:*

$$\begin{cases} \widetilde{q}_1(t) = \dfrac{\widehat{q}_1^*(t) + b_1(t)\widehat{q}_2^*(t)}{1 - b_1(t)b_2(t)}, \\ \widetilde{q}_2(t) = \dfrac{\widehat{q}_2^*(t) + b_2(t)\widehat{q}_1^*(t)}{1 - b_1(t)b_2(t)}, \end{cases} \tag{4.4}$$

*where $b_1(t) = \frac{\lambda\mu_1\mu_2\omega_1 m_1 e^{r(T-t)}}{m_1\sigma_1^2 e^{r(T-t)} + 2\sigma_1^2\Lambda}$, $b_2(t) = \frac{\lambda\mu_1\mu_2\omega_2 m_2 e^{r(T-t)}}{m_2\sigma_2^2 e^{r(T-t)} + 2\sigma_2^2\Lambda}$, $\widehat{q}_1^*(t) = \frac{2\Lambda}{m_1 e^{r(T-t)} + 2\Lambda}$ and $\widehat{q}_2^*(t) = \frac{2\Lambda}{m_2 e^{r(T-t)} + 2\Lambda}$. Then, we have the equilibrium reinsurance strategy*

$$q_k^*(t) = \widetilde{q}_k^*(t) \wedge 1, k = 1, 2. \tag{4.5}$$

*The value function is given by*

$$J_k(t, \widehat{z}_k, l, 1) = -\frac{1}{m_k}\exp\{-m_k \widehat{z}_k e^{r(T-t)} + g_k(t) + f_k(t)l\}, \tag{4.6}$$

*where*

$$\begin{aligned} g_k(t) = \int_t^T \Big\{ \Big[ &(d_k - \omega_k d_j) - (\delta(q_k^*(u)) - \omega_k \delta(q_j^*(u))) \\ &-(q_k^*(u)n_k - \omega_k q_j^*(u)n_j)\Big] \times (-m_k e^{r(T-u)}) \\ &+\frac{1}{2}\Big[\sigma_k^2 q_k^{*2}(u) + \omega_k^2\sigma_j^2 q_j^{*2}(u) - 2\rho\omega_k\sigma_k\sigma_j q_k^*(u)q_j^*(u)\Big] \\ &\times m_k^2 e^{2r(T-u)} + \mathcal{K}\beta f_k(u)\Big\} du \end{aligned} \tag{4.7}$$

*and*

$$f_k(t) = \begin{cases} \dfrac{e^{(\vartheta_1-\vartheta_2)(T-t)} - 1}{e^{(\vartheta_1-\vartheta_2)(T-t)} - \frac{\vartheta_2}{\vartheta_1}}\vartheta_2, & \widehat{\rho} \neq \pm 1, \\ -\dfrac{\alpha^2}{2(\mathcal{K}+\nu\alpha)}[1 - e^{-(\mathcal{K}+\nu\alpha)(T-t)}], & \widehat{\rho} = 1, \\ -\dfrac{\alpha^2}{2(\mathcal{K}-\nu\alpha)}[1 - e^{-(\mathcal{K}-\nu\alpha)(T-t)}], & \widehat{\rho} = -1 \text{ and } \mathcal{K} \neq \nu\alpha, \\ -\dfrac{\alpha^2}{2}(T-t), & \widehat{\rho} = -1 \text{ and } \mathcal{K} = \nu\alpha, \end{cases} \tag{4.8}$$

*where*

$$\vartheta_{1,2} = \frac{\mathcal{K} + \widehat{\rho}\nu\alpha \pm \sqrt{\mathcal{K}^2 + 2\widehat{\rho}\nu\alpha\mathcal{K} + \nu^2\alpha^2}}{2}. \tag{4.9}$$

**Proof.** This proof is similar to that in Theorem 4.2, we omit it. □

Note that the equilibrium investment strategies $\gamma_k^*(t) = 0, k = 1, 2$ because the defaultable bond can not be traded after default. Based on the results of Theorem 4.1, the following subsection will derive the equilibrium strategies and associated value functions for each insurer before default.

### *4.2. Equilibrium strategy before default*

In this subsection, we address the non-zero-sum stochastic differential reinsurance-investment game before default and provide explicit expressions of the pre-default equilibrium strategy and associated value functions.

**Theorem 4.2.** *[Pre-default] For any $t \in [0, \tau \wedge T]$ and $k = 1, 2$, The equilibrium investment strategy $\gamma_k^*(t)$ is given by*

$$\gamma_k^*(t) = \frac{\widehat{\gamma}_k^*(t) + \omega_k \widehat{\gamma}_j^*(t)}{1 - \omega_k\omega_j}, \quad \text{for } k \neq j \in 1, 2, \tag{4.10}$$

*where $\widehat{\gamma}_k^*(t) = \frac{(\ln\frac{1}{\Delta} + \Delta - 1)e^{-\frac{\eta}{\zeta}(T-t)} - \Delta + 1}{m_k\zeta}e^{-r(T-t)}, k = 1, 2.$*

*For the equilibrium investment strategy $\theta_k^*(t)$ and reinsurance strategy $q_k^*(t)$ are given by (4.2) and (4.5), respectively. The value function is then given by*

$$\begin{aligned} J_k(t, \widehat{z}_k, l, 0) &= -\frac{1}{m_k}\exp\{-m_k\widehat{z}_k e^{r(T-t)} + g_k(t) + G_k(t) + f_k(t)l\} \\ &= J_k(t, \widehat{z}_k, l, 1)e^{G_k(t)}, \end{aligned} \tag{4.11}$$

*where*

$$G_k(t) = \Big(\ln\frac{1}{\Delta} + \Delta - 1\Big)e^{-\frac{\eta}{\zeta}(T-t)} - \ln\frac{1}{\Delta} - \Delta + 1. \tag{4.12}$$

**Proof.** When $H(t) = 0$, the HJB Eq. (4.1) becomes:

$$\begin{aligned} 0 = \frac{\partial J_k(t, \widehat{z}_k, l, 0)}{\partial t} &+ \Big\{\Big[r\widehat{z}_k + (d_k - \omega_k d_j) - (\delta(q_k(t)) - \omega_k\delta(q_j^*(t))) \\ &-(q_k(t)n_k - \omega_k q_j^*(t)n_j) + (\theta_k(t) - \omega_k\theta_j^*(t))\alpha l \\ &+(\gamma_k(t) - \omega_k\gamma_j^*(t))\eta\Big]\frac{\partial J_k(t, \widehat{z}_k, l, 0)}{\partial\widehat{z}_k} + \frac{1}{2}\Big[(\theta_k(t) - \omega_k\theta_j^*(t))^2 l \\ &+\sigma_k^2 q_k^2(t) + \omega_k^2\sigma_j^2 q_j^{*2}(t) - 2\rho\omega_k\sigma_k\sigma_j q_k(t)q_j^*(t)\Big] \\ &\times\frac{\partial^2 J_k(t, \widehat{z}_k, l, 0)}{\partial\widehat{z}_k^2} + \mathcal{K}(\beta - l)\frac{\partial J_k(t, \widehat{z}_k, l, 0)}{\partial l} \\ &+\frac{1}{2}\nu^2 l\frac{\partial^2 J_k(t, \widehat{z}_k, l, 0)}{\partial l^2} + \widehat{\rho}(\theta_k(t) - \omega_k\theta_j^*(t))\nu l\frac{\partial^2 J_k(t, \widehat{z}_k, l, 0)}{\partial\widehat{z}_k\partial l} \\ &+\Big(J_k(t, \widehat{z}_k - (\gamma_k(t) - \omega_k\gamma_j^*(t))\zeta, l, 1) - J_k(t, \widehat{z}_k, l, 0)\Big)h^P\Big\} \end{aligned} \tag{4.13}$$

with the boundary condition $J_k(T, \widehat{z}_k, \nu, 0) = -\frac{1}{m_k}e^{-m_k\widehat{z}_k}$. To solve this equation, we conjecture that

$$J_k(t, \widehat{z}_k, l, 0) = -\frac{1}{m_k}\exp\{-m_k\widehat{z}_k e^{r(T-t)} + g_{0k}(t) + f_{0k}(t)l\}, \tag{4.14}$$

where $g_{0k}(t)$, $f_{0k}t$ are two functions to determine later. Then we obtain:

$$\begin{cases} \dfrac{\partial J_k(t, \widehat{z}_k, l, 0)}{\partial t} = J^k(t, \widehat{z}_k, l, 0)(m_k\widehat{z}_k r e^{r(T-t)} + g_{0k}'(t) + f_{0k}'(t)l), \\ \dfrac{\partial J_k(t, \widehat{z}_k, l, 0)}{\partial\widehat{z}_k} = J^k(t, \widehat{z}_k, l, 0)(-m_k e^{r(T-t)}), \\ \dfrac{\partial^2 J_k(t, \widehat{z}_k, l, 0)}{\partial\widehat{z}_k^2} = J^k(t, \widehat{z}_k, l, 0)(m_k^2 e^{2r(T-t)}), \\ \dfrac{\partial J_k(t, \widehat{z}_k, l, 0)}{\partial l} = f_{0k}(t)J^k(t, \widehat{z}_k, l, 0), \\ \dfrac{\partial^2 J_k(t, \widehat{z}_k, l, 0)}{\partial l^2} = f_{0k}^2(t)J^k(t, \widehat{z}_k, l, 0), \\ \dfrac{\partial^2 J_k(t, \widehat{z}_k, l, 0)}{\partial\widehat{z}_k\partial l} = -m_k e^{r(T-t)}f_{0k}(t)J^k(t, \widehat{z}_k, l, 0). \end{cases} \tag{4.15}$$

Inserting (4.15) into the HJB Eq. (4.13) leads to the following differential equation:

$$
\begin{aligned}
0 = {} & g'_{0k}(t) + f'_{0k}(t)l + \mathcal{K}(\beta - l)f_{0k}(t) + \frac{1}{2}\nu^2 l f^2_{0k}(t) \\
& + \inf_{q_k(t)} \Bigg\{ \Big[ (d_k - \omega_k d_j) - (\delta(q_k(t)) - \omega_k \delta(q^*_j(t))) \\
& -(q_k(t)n_k - \omega_k q^*_j(t)n_j) \Big] (-m_k e^{r(T-t)}) \\
& + \frac{1}{2}\Big[ \sigma_k^2 q_k^2(t) + \omega_k^2 \sigma_j^2 q_j^{*2}(t) - 2\rho\omega_k\sigma_k\sigma_j q_k(t) q^*_j(t) \Big] \\
& \times m_k^2 e^{2r(T-t)} \Bigg\} + \inf_{\theta_k(t)} \Bigg\{ (\theta_k(t) - \omega_k\theta^*_j(t))\alpha l(-m_k e^{r(T-t)}) \\
& + \frac{1}{2}\Big[ (\theta_k(t) - \omega_k\theta^*_j(t))^2 l m_k^2 e^{2r(T-t)} - 2f_{0k}(t) \\
& \times (\theta_k(t) - \omega_k\theta^*_j)\widehat{\rho}\nu l m_k e^{r(T-t)} \Big] \Bigg\} \\
& + \inf_{\gamma_k(t)} \Bigg\{ (\gamma_k(t) - \omega_k\gamma^*_j(t))\eta(-m_k e^{r(T-t)}) \\
& + \Big( e^{m_k(\gamma_k(t) - \omega_k\gamma^*_j(t))\zeta e^{r(T-t)} + (g_k(t) - g_{0k}(t)) + (f_k(t) - f_{0k}(t))l} - 1 \Big) \times h^P \Bigg\}.
\end{aligned} \tag{4.16}
$$

Using the first-order conditions for a regular interior minimizer of (4.16), we have

$$
\begin{cases}
q^*_1(t) = \left( \dfrac{2\Lambda}{m_1 e^{r(T-t)} + 2\Lambda} + \dfrac{\rho\omega_1\sigma_2 m_1 e^{r(T-t)}}{m_1\sigma_1 e^{r(T-t)} + 2\sigma_1\Lambda} q^*_2(t) \right) \wedge 1, \\
q^*_2(t) = \left( \dfrac{2\Lambda}{m_2 e^{r(T-t)} + 2\Lambda} + \dfrac{\rho\omega_2\sigma_1 m_2 e^{r(T-t)}}{m_2\sigma_2 e^{r(T-t)} + 2\sigma_2\Lambda} q^*_1(t) \right) \wedge 1,
\end{cases} \tag{4.17}
$$

$$
\begin{cases}
\theta^*_1(t) = \left( \dfrac{\alpha}{m_1} + \dfrac{f_{01}(t)\widehat{\rho}\nu}{m_1} \right) e^{-r(T-t)} + \omega_1\theta^*_2(t), \\
\theta^*_2(t) = \left( \dfrac{\alpha}{m_2} + \dfrac{f_{02}(t)\widehat{\rho}\nu}{m_2} \right) e^{-r(T-t)} + \omega_2\theta^*_1(t),
\end{cases} \tag{4.18}
$$

and

$$
\begin{cases}
\gamma^*_1(t) = \dfrac{\ln\frac{1}{\Delta} + (g_{01}(t) - g_1(t)) + (f_{01}(t) - f_1(t))l}{m_1\zeta} e^{-r(T-t)} + \omega_1\gamma^*_2(t), \\
\gamma^*_2(t) = \dfrac{\ln\frac{1}{\Delta} + (g_{02}(t) - g_2(t)) + (f_{02}(t) - f_2(t))l}{m_2\zeta} e^{-r(T-t)} + \omega_2\gamma^*_1(t).
\end{cases} \tag{4.19}
$$

We can derive the equilibrium strategies $\theta^*_k(t)$ (resp. $\gamma^*_k(t)$) in (4.2) (resp. (4.10)) in a simple and straightforward manner. Moreover, from the system of Eq. (4.17), we find that the equilibrium reinsurance strategy before default is the same as that after default in Theorem 4.1.

For the equilibrium reinsurance strategies $q^*_k(t)$, we first define $\widetilde{q}_k(t)$ as (4.4), which is the solution of the following system of equations:

$$
\begin{cases}
\widetilde{q}_1(t) = \dfrac{2\Lambda}{m_1 e^{r(T-t)} + 2\Lambda} + \dfrac{\rho\omega_1\sigma_2 m_1 e^{r(T-t)}}{m_1\sigma_1 e^{r(T-t)} + 2\sigma_1\Lambda} \widetilde{q}_2(t), \\
\widetilde{q}_2(t) = \dfrac{2\Lambda}{m_2 e^{r(T-t)} + 2\Lambda} + \dfrac{\rho\omega_2\sigma_1 m_2 e^{r(T-t)}}{m_2\sigma_2 e^{r(T-t)} + 2\sigma_2\Lambda} \widetilde{q}_1(t).
\end{cases}
$$

Use the fact that $1 \geq \omega_k \geq 0$, $\sigma_k > 0$, $\Lambda > 0$, $m_k > 0$, $1 \geq \rho \geq 0$, and derive that $\widetilde{q}_k(t) \geq 0$ because

$$
1 - b_1(t)b_2(t) = 1 - \frac{\rho^2\omega_1\omega_2\sigma_1\sigma_2 m_1 m_2 e^{2r(T-t)}}{(m_1\sigma_1 e^{r(T-t)} + 2\sigma_1\Lambda)(m_2\sigma_2 e^{r(T-t)} + 2\sigma_2\Lambda)} \geq 0.
$$

Then, we can obtain the following equilibrium reinsurance strategies in four cases.

If $\widetilde{q}_k(t) \leq 1, k = 1, 2,$ $q^*_k(t) = \widetilde{q}_k(t)$.

If $\widetilde{q}_1(t) \leq 1$ and $\widetilde{q}_2(t) > 1$, we have $q^*_2(t) = 1$. From system of Eq. (4.17), we have that $q^*_1(t) = \frac{2\Lambda}{m_1 e^{r(T-t)} + 2\Lambda} + \frac{\rho\omega_1\sigma_2 m_1 e^{r(T-t)}}{m_1\sigma_1 e^{r(T-t)} + 2\sigma_1\Lambda} = \frac{2\Lambda}{m_1 e^{r(T-t)} + 2\Lambda} + \frac{\lambda\mu_1\mu_2\omega_1 m_1 e^{r(T-t)}}{m_1\sigma_1^2 e^{r(T-t)} + 2\sigma_1^2\Lambda}$.

Similarly, if $\widetilde{q}_1(t) > 1$ and $\widetilde{q}_2(t) \leq 1$, it indicate that $(q^*_1(t), q^*_2(t)) = \left(1, \frac{2\Lambda}{m_2 e^{r(T-t)} + 2\Lambda} + \frac{\lambda\mu_1\mu_2\omega_2 m_2 e^{r(T-t)}}{m_2\sigma_2^2 e^{r(T-t)} + 2\sigma_2^2\Lambda}\right)$.

If $\widetilde{q}_k(t) > 1, k = 1, 2,$ we have $q^*_1(t) = 1$ and $q^*_2(t) = 1$. Therefor, we can conclude that

$$
q^*_k(t) = \widetilde{q}^*_k(t) \wedge 1, \quad k = 1, 2.
$$

Substituting these equilibrium rules into (4.16), we obtain

$$
\begin{aligned}
0 = {} & g'_{0k}(t) + \Big[ (d_k - \omega_k d_j) - (\delta(q^*_k(t)) - \omega_k\delta(q^*_j(t))) \\
& -(q^*_k(t)n_k - \omega_k q^*_j(t)n_j) \Big] \times (-m_k e^{r(T-t)}) \\
& + \frac{1}{2}\Big[ \sigma_k^2 q_k^{*2}(t) + \omega_k^2\sigma_j^2 q_j^{*2}(t) - 2\rho\omega_k\sigma_k\sigma_j q^*_k(t) q^*_j(t) \Big] m_k^2 e^{2r(T-t)} \\
& + \mathcal{K}\beta f_{0k}(t) - \frac{\eta}{\zeta}\ln\frac{1}{\Delta} + h^P\Big(\frac{1}{\Delta} - 1\Big) + \frac{g_k(t) - g_{0k}(t)}{\zeta}\eta \\
& + \Big[ f'_{0k}(t) + \frac{1}{2}\nu^2(1 - \widehat{\rho}^2) f^2_{0k}(t) - (\widehat{\rho}\alpha\nu + \mathcal{K}) f_{0k}(t) \\
& + \frac{f_k(t) - f_{0k}(t)}{\zeta}\eta - \frac{\alpha^2}{2} \Big] l.
\end{aligned} \tag{4.20}
$$

We can divide (4.20) into the following two differential equations:

$$
\begin{aligned}
& g'_{0k}(t) - \frac{g_{0k}(t)}{\zeta}\eta + \Big[ (d_k - \omega_k d_j) - (\delta(q^*_k(t)) - \omega_k\delta(q^*_j(t))) \\
& -(q^*_k(t)n_k - \omega_k q^*_j(t)n_j) \Big] \times (-m_k e^{r(T-t)}) \\
& + \frac{1}{2}\Big[ \sigma_k^2 q_k^{*2}(t) + \omega_k^2\sigma_j^2 q_j^{*2}(t) - 2\rho\omega_k\sigma_k\sigma_j q^*_k(t) q^*_j(t) \Big] m_k^2 e^{2r(T-t)} \\
& + \mathcal{K}\beta f_{0k}(t) - \frac{\eta}{\zeta}\ln\frac{1}{\Delta} + h^P\Big(\frac{1}{\Delta} - 1\Big) + \frac{g_k(t)}{\zeta}\eta = 0,
\end{aligned} \tag{4.21}
$$

$$
\begin{aligned}
& f'_{0k}(t) + \frac{1}{2}\nu^2(1 - \widehat{\rho}^2) f^2_{0k}(t) - \Big( \widehat{\rho}\alpha\nu + \mathcal{K} + \frac{\eta}{\zeta} \Big) f_{0k}(t) \\
& + \frac{\eta}{\zeta} f_k(t) - \frac{\alpha^2}{2} = 0,
\end{aligned} \tag{4.22}
$$

with the boundary condition $g_{0k}(T) = f_{0k}(T) = 0$.

Let $G_k(t) = g_{0k}(t) - g_k(t), k = 1, 2,$ and $G_k(t)$ is differentiated w.r.t. $t$, we obtain:

$$
G'_k(t) = g'_{0k}(t) - g'_k(t) = \frac{\eta}{\zeta}G_k(t) + \frac{\eta}{\zeta}\ln\frac{1}{\Delta} - h^P\Big(\frac{1}{\Delta} - 1\Big). \tag{4.23}
$$

Because $G_k(T) = g_{0k}(T) - g_k(T) = 0$, we have

$$
G_k(t) = (\ln\frac{1}{\Delta} + \Delta - 1)e^{-\frac{\eta}{\zeta}(T-t)} - \ln\frac{1}{\Delta} - \Delta + 1. \tag{4.24}
$$

Applying Lemma 3.1 and Corollary 3.1 reported by Zhu et al. (2015), we can solve the nonlinear Riccati differential Eq. (4.22), whose solution is $f_{0k}(t) = f_k(t)$, as presented in (4.8). □

Combining Theorems 4.1 and 4.2, we obtain the following result directly.

**Theorem 4.3.** *Given the CARA preferences, the value functions for insurer $k = 1, 2$, are given by*

$$
\begin{aligned}
J_k(t, \widehat{z}_k, l, h) = {} & -\frac{1}{m_k}\exp\{-m_k\widehat{z}_k e^{r(T-t)} + g_k(t) \\
& + G_k(t)(1 - h) + f_k(t)l\},
\end{aligned} \tag{4.25}
$$

*where $g_k(t)$, $f_k(t)$ and $G_k(t)$ are given in (4.7), (4.8), and (4.24), respectively. The optimal reinsurance-investment strategy for each insurers are triple process $\pi_k^*(t) = (q_k^*(t), \theta_k^*(t), \gamma_k^*(t))$, where $\theta_k^*(t)$ and $q_k^*(t)$ are given in (4.2) and (4.4), and the equilibrium investment strategy $\gamma_k^*(t)$ is*

$$\gamma_k^*(t) = \begin{cases} \dfrac{\widehat{\gamma}_k^*(t) + \omega_k \widehat{\gamma}_j^*(t)}{1-\omega_k\omega_j}, & \text{for } t \in [0, \tau\wedge T), \\ 0, & \text{for } t \in [\tau \wedge T, T]. \end{cases} \tag{4.26}$$

**Remark 4.1.** Note that the HJB Eq. (4.13) that is associated with the pre-default value function $J_k(t, \widehat{z}_k, l, 0)$ also depends on the post-default value function $J_k(t, \widehat{z}_k, l, 1)$.

**Remark 4.2.** Regardless of whether the corporate bond defaults, the equilibrium reinsurance strategy $q_k^*(t)$ and the equilibrium investment strategy $\theta_k^*(t)$ do not change. In other words, the insurer moves the recovered value from the default bond into the risk-free asset after the default of the ZCB.[3] This is due to the setting that either the surplus process of each insurer or the stock price process is uncorrelated with the corporate bond's price process. Moreover, under the CARA utility each insurer is optimal to invest in the defaultable bond if the risk premium $\frac{1}{\Delta} > 0$. As a result, the post-default value function $J_k(t, \hat{z}_k, l, 1)$ is smaller than the pre-default value function $J_k(t, \hat{z}_k, l, 0)$, and the $(e^{G_k(t)} - 1)J_k(t, \hat{z}_k, l, 1)$ is the additional income due to the investment in the defaultable bond.

**Remark 4.3.** When $\omega_1 = \omega_2 = 0$, the equilibrium strategy $\pi_k^*(t) = (q_k^*(t), \theta_k^*(t), \gamma_k^*(t))$ is reduced to the *regular strategy* $\widehat{\pi}_k^*(t) = (\widehat{q}_k^*(t), \widehat{\theta}_k^*(t), \widehat{\gamma}_k^*(t))$, which is optimal for the case without the relative performance concern.

**Corollary 4.1.** *(Best response to competition) If $\omega_2 = 0$ and $\omega_1 > 0$ (i.e., insurer 1 faces no competition from insurer 2), then the Nash equilibrium reinsurance-investment strategies are given as follow:*

$$\begin{cases} q_1^*(t) = \Big(\widehat{q}_1^*(t) + b_1(t)\widehat{q}_2^*(t)\Big)\wedge 1, & \text{for } t \in [0,T], \\ q_2^*(t) = \widehat{q}_2^*(t), & \text{for } t \in [0,T], \\ \theta_1^*(t) = \widehat{\theta}_1^*(t) + \omega_1\widehat{\theta}_2^*(t), & \text{for } t \in [0,T], \\ \theta_2^*(t) = \widehat{\theta}_2^*(t), & \text{for } t \in [0,T], \\ \gamma_1^*(t) = \begin{cases} \widehat{\gamma}_1^*(t) + \omega_1\widehat{\gamma}_2^*(t), & \text{for } t \in [0, \tau\wedge T], \\ 0, & \text{for } t \in [\tau\wedge T, T] \end{cases} \\ \gamma_2^*(t) = \begin{cases} \widehat{\gamma}_2^*(t), & \text{for } t \in [0, \tau\wedge T], \\ 0, & \text{for } t \in [\tau\wedge T, T]. \end{cases} \end{cases} \tag{4.27}$$

*Corollary 4.1 shows that insurer 2's equilibrium reinsurance-investment strategy $\pi_2^*(t) = (q_2^*(t), \theta_2^*(t), \gamma_2^*(t))$ is consistent with the regular strategy $\widehat{\pi}_2^*(t) = (\widehat{q}_2^*(t), \widehat{\theta}_2^*(t), \widehat{\gamma}_2^*(t))$, whereas insurer 1's equilibrium reinsurance-investment strategy $\pi_1^*(t)$ can be divided into two parts. The first part is the regular strategy $\widehat{\pi}_1^*(t)$, which is likely due to the partial objective of maximizing the terminal wealth; the second part $(b_1(t)q_2^*(t),\ \omega_1\widehat{\theta}_2^*(t),\ \omega_1\widehat{\gamma}_2^*(t))$ is induced by the relative performance concern. Specifically, the presence of the relative concern affects insurer 1's reinsurance decision and results in a riskier reinsurance policy.*

*Insurer 1's regular reinsurance strategy $\widehat{q}_1^*(t) = \frac{2\Lambda}{m_1 e^{-r(T-t)} + 2\Lambda}$ decreases in the risk aversion parameter $m_1$, or $1 - \widehat{q}_1^*(t)$ increases in $m_1$. This indicates that a more risk averse insurer will buy more reinsurance contracts. By contrast, in a competitive environment, the competitive coefficient $b_1(t)$ increases in $m_1$. As a result, the equilibrium reinsurance strategy $q_1^*(t)$ may increase in $m_1$, depending on insurer 2's regular reinsurance policy $\widehat{q}_2^*(t)$. Thus, a more risk averse insurer may buy less reinsurance contracts in the presence of competition. In particular, $q_1^*(t) = \frac{\rho\omega_1\sigma_2}{\sigma_1}\widehat{q}_2^*(t) \wedge 1 > 0$ as $m_1 \uparrow +\infty$, that is, an extreme risk averse insurer may still exposure to the claim risk. In addition, in this special case, insurer 1's equilibrium reinsurance strategy is simply to mimic the optimal strategy $\widehat{q}_2^*(t)$ that is followed by insurer 2; this reinsurance strategy decreases in insurer 2's risk aversion parameter $m_2$. These new features of the equilibrium strategies should be induced by the effect of the relative performance concerns that distort the rational reinsurance decisions of the insurers. This point is further affirmed after we observe that the equilibrium strategies are actually riskier than the regular strategies. The extra terms in the equilibrium strategies of insurer 1 demonstrate insurer 1's response to the competition: to outperform insurer 2, it takes more aggressive reinsurance and investment strategies. Next, we will perform a detailed analysis of the equilibrium reinsurance-investment strategy in a more general case of competition when both insurers have relative concerns ($\omega_k > 0, k = 1, 2$).*

**Table 1**
Insurer $k$'s equilibrium reinsurance strategy $q_k^*(t)$.

| $\partial q_k^*(t)/\partial\omega_k$ | $\partial q_k^*(t)/\partial\omega_j$ | $\partial q_k^*(t)/\partial\mu_k$ | $\partial q_k^*(t)/\partial\mu_j$ | $\partial q_k^*(t)/\partial\sigma_k$ | $\partial q_k^*(t)/\partial\sigma_j$ |
|---|---|---|---|---|---|
| + | + | + | + | − | − |

**Corollary 4.2.** *If $\omega_k > 0, k = 1, 2$, the Nash equilibrium strategy $\pi_k^*(t) = (q_k^*(t), \theta_k^*(t), \gamma_k^*(t))$ has the following properties:*

*(I) Insurer $k$ increases its equilibrium proportional reinsurance strategy $q_k^*(t)$ relative to the regular reinsurance strategy $\widehat{q}_k^*(t)$(i.e., $\omega_k = 0, k = 1, 2$) without competition. The sensitivities of insurer k's equilibrium reinsurance strategy with respect to the parameters are given in Table 1.*

*(II) If the equity premium is strictly positive, then insurer $k$ will hold a positive position of the stock, and the investment($\theta_k^*(t)$) will be larger relative to the regular strategy without competition ($\omega_k = 0, k = 1, 2$). The sensitivities of the equilibrium investment strategy $\theta_k^*(t)$ are summarized in Table 2.*

*(III) Each insurer will choose to increase their investment in the corporate bond relative to the case of no competition ($\omega_k = 0, k = 1, 2$), and each insurer will always hold a positive position of the corporate bond with a positive risk premium (i.e., $\gamma_k^*(t) > 0$, if $\frac{1}{\Delta} > 1$); whereas $\gamma_k^*(t) = 0$ and $J_k(t, \widehat{z}_k, l, 1) = J_k(t, \widehat{z}_k, l, 0)$ if $\frac{1}{\Delta} = 1$. Moreover, we have the following sensitivity analyses in Table 3.*

**Proof.** (I). From Theorems 4.1 and 4.2, we know that $q_k^*(t) = \frac{b_k(t)\widehat{q}_j^*(t) + \widehat{q}_k^*(t)}{1 - b_1(t)b_2(t)} \wedge 1$ for $k \neq j \in \{1, 2\}$. Because $\widehat{q}_k^*(t) > 0, b_k(t) > 0$, and $1 > 1 - b_1(t)b_2(t) > 0$, we have $q_k^*(t) \geq \widehat{q}_k^*(t)$. This step implies that the insurer purchases fewer reinsurance contracts. We can easily derive the relationship between the equilibrium reinsurance strategy and model parameters; and the details of this procedure are omitted here.

(II). To prepare for proof (II), we need to verify that $f_k(t) \leq 0$. From the definition of $f_k$ in Theorems 4.1 and 4.2, we can derive $f_k(t) \leq f_k(T) = 0$ straightforwardly.

Let $u = \hat{\rho}\nu f_k(t)$. Using Eq. (4.22), we can show that the function $u$ satisfies the following equation:

$$u' + \frac{\nu}{2\hat{\rho}}(1 - \hat{\rho}^2)u^2 - (\hat{\rho}\alpha\nu + \mathcal{K})u - \frac{\alpha^2\nu\hat{\rho}}{2} = 0, \tag{4.28}$$

[3] We thank an anonymous referee for pointing out this. In Bielecki and Jang (2006), the optimal investment strategy also does not change before or after the default of the defaultable bond.

**Table 2**
Insurer $k$'s equilibrium investment strategy $\theta_k^*(t)$.

| $\partial\theta_k^*(t)/\partial\omega_k$ | $\partial\theta_k^*(t)/\partial\omega_j$ | $\partial\theta_k^*(t)/\partial m_k$ | $\partial\theta_k^*(t)/\partial m_j$ | $\partial\theta_k^*(t)/\partial\hat{\rho}$ | $\partial\theta_k^*(t)/\partial\mathcal{K}$ | $\partial\theta_k^*(t)/\partial\alpha$ |
|---|---|---|---|---|---|---|
| + | + | − | − | − | $+(\hat{\rho}>0)-(\hat{\rho}<0)$ | + |

**Table 3**
Insurer $k$'s equilibrium investment strategy $\gamma_k^*(t)$.

| $\partial\gamma_k^*(t)/\partial\omega_k$ | $\partial\gamma_k^*(t)/\partial\omega_j$ | $\partial\gamma_k^*(t)/\partial\frac{1}{\Delta}$ | $\partial\gamma_k^*(t)/\partial\zeta$ | $\partial\gamma_k^*(t)/\partial m_k$ | $\partial\gamma_k^*(t)/\partial m_j$ |
|---|---|---|---|---|---|
| + | + | + | − | − | − |

where $u'=\partial u/\partial t$. Differentiating Eq. (4.28) w.r.t. to $\hat{\rho}$, we obtain the following equation for the derivative $u_{\hat{\rho}}$ of $u$ w.r.t. to $\hat{\rho}$:

$$\frac{\partial u'}{\partial\hat{\rho}}-\frac{\nu}{2}(\frac{1}{\hat{\rho}^2}+1)u^2+\frac{\nu}{\hat{\rho}}(1-\hat{\rho}^2)u\frac{\partial u}{\partial\hat{\rho}}-(\hat{\rho}\alpha\nu+\mathcal{K})\frac{\partial u}{\partial\hat{\rho}}$$
$$-\alpha\nu u-\frac{\alpha^2\nu}{2}=0. \tag{4.29}$$

Let $K=\frac{\nu}{\hat{\rho}}(1-\hat{\rho}^2)u-(\hat{\rho}\alpha\nu+\mathcal{K})$; then, we can rewrite (4.29) as

$$\left(\frac{\partial u}{\partial\hat{\rho}}\right)'+K\frac{\partial u}{\partial\hat{\rho}}-\frac{\nu}{2}u^2-\frac{\nu}{2}(u+\alpha)^2=0.$$

The solution to this ordinary equation is given by

$$\frac{\partial u}{\partial\hat{\rho}}=-e^{\int_t^T Kds}\int_t^T e^{-\int_s^T Kdv}\left[\frac{\nu}{2}u^2+\frac{\nu}{2}(u+\alpha)^2\right]ds<0,$$

which indicates that $u$ decreases as $\hat{\rho}$ increases. Because $\widehat{\theta}_k^*(t)=\frac{(\alpha+\hat{\rho}\nu f_k(t))}{m_k}e^{-r(T-t)}=\frac{(\alpha+u(\hat{\rho}))}{m_k}e^{-r(T-t)}$, $\widehat{\theta}_k^*(t)$ is a decreasing function of $\hat{\rho}$. Additionally, we can consider $\widehat{\theta}_k^*(t)$ to be a function of $\hat{\rho}$, and for a fixed $t$, $\widehat{\theta}_k^*(t;\hat{\rho}):=\widehat{\theta}_k^*(t)$. Then, $\widehat{\theta}_k^*(t;\hat{\rho})\geq\widehat{\theta}_k^*(t;1^-)=\lim_{\hat{\rho}\uparrow 1}\frac{(\alpha+u)}{m_k}e^{-r(T-t)}$ and:

$$\lim_{\hat{\rho}\uparrow 1}\frac{(\alpha+u)}{m_k}e^{-r(T-t)}=\frac{1}{m_k}(\alpha+\lim_{\hat{\rho}\uparrow 1}\nu f_k(t))e^{-r(T-t)}.$$

From (4.8), we have $\lim_{\hat{\rho}\uparrow 1}\nu f_k(t)=0$. Then, $\widehat{\theta}_k^*(t;\hat{\rho})\geq\lim_{\hat{\rho}\uparrow 1}\widehat{\theta}_k^*(t;1^-)=\frac{\alpha}{m_k}e^{-r(T-t)}>0$.

For $\hat{\rho}=1$,

$$\begin{aligned}\widehat{\theta}_k^*(t;\hat{\rho})\Big|_{\hat{\rho}=1}&=\frac{1}{m_k}\alpha\left(1-\frac{\alpha\nu}{2(\mathcal{K}+\nu\alpha)}+\frac{\alpha\nu}{2(\mathcal{K}+\nu\alpha)}e^{-(\mathcal{K}+\nu\alpha)(T-t)}\right)\\&=\frac{1}{m_k}\alpha\left(\frac{2\mathcal{K}+\alpha\nu}{2(\mathcal{K}+\nu\alpha)}+\frac{\alpha\nu}{2(\mathcal{K}+\nu\alpha)}e^{-(\mathcal{K}+\nu\alpha)(T-t)}\right).\end{aligned}$$

Because $\mathcal{K}>0,\alpha>0$ and $\nu>0$, $\widehat{\theta}_k^*(t;\hat{\rho})\Big|_{\hat{\rho}=1}>0$. Based on these results, we can conclude that $\widehat{\theta}_k^*(t;\hat{\rho}):=\widehat{\theta}_k^*(t)>0$ for $\forall t\in[0,T]$. The following results are then true:

(1) $\theta_k^*(t)=\frac{\omega_k\widehat{\theta}_j^*(t)+\widehat{\theta}_k^*(t)}{1-\omega_1\omega_2}\geq\widehat{\theta}_k^*(t)$;
(2) $\frac{\partial\theta_k^*(t)}{\partial m_k}<0$ and $\frac{\partial\theta_k^*(t)}{\partial m_j}<0$;
(3) $\frac{\partial\theta_k^*(t)}{\partial\omega_k}>0$ and $\frac{\partial\theta_k^*(t)}{\partial\omega_j}>0$.

Differentiating Eq. (4.28) with respect to $\mathcal{K}$, we can obtain the following equation for the derivative $u_{\mathcal{K}}$ of $u$ w.r.t. $\mathcal{K}$:

$$\frac{\partial u'}{\partial\mathcal{K}}+\frac{\nu}{\hat{\rho}}(1-\hat{\rho}^2)u\frac{\partial u}{\partial\mathcal{K}}-(\hat{\rho}\alpha\nu+\mathcal{K})\frac{\partial u}{\partial\mathcal{K}}-u=0,$$

which is equivalent to

$$\left(\frac{\partial u}{\partial\mathcal{K}}\right)'+Ku_{\mathcal{K}}-u=0.$$

Then, we can derive the solution to the above ordinary equation as

$$\frac{\partial u}{\partial\mathcal{K}}=-e^{\int_t^T Kds}\int_t^T e^{-\int_s^T Kdv}uds,$$

for $\hat{\rho}>0$, $u_{\mathcal{K}}>0$, and $u_{\mathcal{K}}<0$ if $\hat{\rho}<0$. Therefore, the equilibrium investment strategy $\theta_k^*(t)$ increases as $\mathcal{K}$ increases if $\hat{\rho}>0$ and decreases as $\mathcal{K}$ increases if $\hat{\rho}<0$.

Let $u=\alpha+\hat{\rho}\nu f_k(t)$; then, we obtain:

$$u'+\frac{\nu}{2\hat{\rho}}(1-\hat{\rho}^2)(u^2-2\alpha u+\alpha^2)-(\hat{\rho}\nu\alpha+\mathcal{K})(u-\alpha)-\frac{\alpha^2\hat{\rho}\nu}{2}=0. \tag{4.30}$$

Differentiating Eq. (4.30) w.r.t. $\alpha$, we obtain the following equation for the derivation $u_\alpha$ of $u$ w.r.t. $\alpha$:

$$\left(\frac{\partial u}{\partial\alpha}\right)'-\frac{\nu}{\hat{\rho}}(1-\hat{\rho}^2)u+\frac{\nu}{\hat{\rho}}\alpha+\mathcal{K}=0,$$

where $\hat{K}=\frac{\nu}{\hat{\rho}}(1-\hat{\rho}^2)u-(\frac{\nu}{\hat{\rho}}\alpha+\mathcal{K})$. Thus,

$$\frac{\partial u}{\partial\alpha}=-e^{\int_t^T\hat{K}ds}\int_t^T e^{-\int_s^T\hat{K}dv}\left(-\frac{\nu}{\hat{\rho}}(u-\alpha)+\mathcal{K}\right)ds.$$

Because $-\frac{\nu}{\hat{\rho}}(u-\alpha)=-\frac{\nu}{\hat{\rho}}\nu\hat{\rho}f_k(t)=-\nu^2 f_k(t)>0$, we have $\partial u/\partial\alpha>0$. Consequently, we can easily derive that $\partial\theta_k^*(t)/\partial\alpha>0$ and $\theta_k^*(t)>0$ if $\alpha>0$.

(III) First, if $\frac{1}{\Delta}=\frac{h^Q}{h^P}=1$, then the optimal result is $\hat{\gamma}_k^*(t)=0, k=1,2$. Thus, the equilibrium investment strategy is described by $\gamma_k^*(t)=\frac{\omega_k\hat{\gamma}_j^*(t)+\hat{\gamma}_k^*(t)}{1-\omega_k\omega_j}=0$.

Similarly, when $\frac{1}{\Delta}=\frac{h^Q}{h^P}>1$, the regular investment strategy $\hat{\gamma}_k^*(t)$ is described by

$$\hat{\gamma}_k^*(t)=\frac{\ln\frac{1}{\Delta}e^{-\frac{\delta}{\zeta}(T-t)}+(1-e^{-\frac{\delta}{\zeta}(T-t)})(1-\Delta)}{m_k\zeta}e^{-r(T-t)}.$$

It is shown that $\hat{\gamma}_k^*(t)>0$, and thus the equilibrium investment strategy is described by $\gamma_k^*(t)>0$.

Using the above results, we can prove the monotone property between the equilibrium investment strategy and model parameters. □

Corollary 4.2 provides an explicit characterization of the equilibrium reinsurance-investment strategy in the general case of competition when both insurers have relative concerns ($\omega_k>0, k=1,2$).

First, as in Corollary 4.2 (I), insurer $k$'s equilibrium reinsurance strategy $q_k^*(t)$ is also larger than the regular strategy without competition; this finding agrees with Corollary 4.1. Additionally, Table 1 shows that $q_k^*(t)$ increases as $\omega_k$ or $\omega_m$ increase; thus, a larger $\omega_k$ implies that insurer $k$ will consider the performance of its opponent more. Therefore, each insurer tends to increase the reinsurance strategy $q_k^*$ and takes a riskier decision to maximize the difference between their terminal wealth. Similarly, a larger $\omega_m$ indicates that insurer $k$ faces stronger competition from its opponent(i.e., insurer $m$); it is shown that increasing the reinsurance strategy $q_k^*$ is the best response to competition. The equilibrium reinsurance strategy also decreases as the volatility parameter $\sigma_k$ increases.

Second, although the equilibrium investment strategy $\theta_k^*(t)$ has a complex form, we provide the following properties for $\theta_k^*(t)$ with a rigorous proof, which is not provided in the existings literature. First, Corollary 4.2 (II) shows a notable result that each insurer's stock holding is always positive if the stock risk premium is positive, and the holding decreases as the risk aversion of each insurer increases; this result is different from that of Liu (2007), in which a CRRA investor is assumed and this monotonicity does not form completely. The equilibrium investment strategy also increases as the competition parameters $\omega_k, k = 1, 2$, increase. Then, as in Corollary 4.1, each insurer will assume a more aggressive investment strategy in a more general setting. Finally, we obtain the relationship between $\theta_k^*(t)$ and the various stock price model parameters in Table 2.

Corollary 4.2(III) shows that insurers will also hold a positive amount of the corporate bonds when the risk premium is positive; the holdings decrease as the risk aversion parameter $m_k$ increases. Each insurer also takes a more aggressive investment decision in this general case, and buys more corporate bonds when $\omega_k$ or $\omega_m$ is larger. Table 3 describes the dependence of the equilibrium investment policy $\gamma_k^*(t)$ on the other parameters.

The following result demonstrates the relationship between the pre- and the post-default equilibrium value functions.

**Corollary 4.3.** *(i) For $k\in\{1, 2\}$ and $h\in\{0, 1\}$, the equilibrium value function $J_k(t, \widehat{z}_k, l, h)$ increases as $\widehat{z}_k$ and $l$ increase.*

*(ii) The pre-default equilibrium value function $J_k(t, \widehat{z}_k, l, 0)$ is always greater than the post-default equilibrium value function $J_k(t, \widehat{z}_k, l, 1)$.*

**Proof.** From the expression of $J_k(t, \widehat{z}_k, l, h)$ in Theorem 4.3, we can obtain these results. A detailed proof is omitted. □

**Remark 4.4.** Since $\widehat{z}_k = z_k - \omega_k z_j$, the equilibrium value function $J_k(t, \widehat{z}_k, l, h)$ decreases in $\omega_k$ or $z_j$, and increases in $z_k$. It implies that the competition will reduce insurers' utilities, and the more wealth insurer $j$ has, the lower utility insurer $k$ $(\neq j)$ achieves. In addition, the value function is concave with respect to the relative wealth.

### *4.3. Verification theorem*

In the dynamic programming approach, a verification lemma is necessary to guarantee that a solution to the HJB equation coincides with the value function. We should verify that the smooth candidate solution derived in the previous section is indeed the value function of this optimization problem. Zeng and Taksar (2013) assume the value function be a concavity function (every convex function is locally Lipschitz), and provide a verification result for a dynamic portfolio optimization problem with a Heston model corresponding to the CRRA utility cases. In a similar case (locally Lipschitz), we apply Zeng and Taksar's result (Lemma A.2) and derive the verification theorem for a reinsurance-investment optimization problem with a Heston model corresponding to the CARA utility cases.

**Theorem 4.4.** *[Verification theorem]For $k = 1, 2$, let $F^k$ be a concave solution to HJB Eq. (4.1) and an integrable function at every stopping-time $\tau_i\in[0, T]$; additionally, let $\pi_k^*$ be described in Theorem 4.3. Then, $\pi_k^*$ is an optimal control for problem (3.7), and $F^k$ is the corresponding value function.*

**Proof.** Let $\mathcal{M} = \mathbb{R}\times\mathbb{R}^+\times\{0, 1\}$. Take a sequence of bounded open sets $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \ldots,$ with $\mathcal{M}_i\subset\mathcal{M}_{i+1}\subset\mathcal{M}, i = 1, 2, \ldots,$ and $\mathcal{M} = \cup_i\mathcal{M}_i$. For $(\widehat{z}_k, l, h)\in\mathcal{M}_1$, let $\tau_i$ be the exit time of $(\widehat{Z}_k^{\pi_k}(t), L(t), H(t))$ from $\mathcal{M}_i$. Then, $\tau_i{\wedge}T\to T$, a.s., as $i\to\infty$.

Applying Itô's lemma to $F^k$ yields

$$\begin{aligned}
&F^k(\tau_i\wedge T, \widehat{Z}_k^{\pi_k}(\tau_i\wedge T), L(\tau_i\wedge T), H(\tau_i\wedge T))\\
&= F^k(t, \widehat{z}_k, l, h) + \int_t^{\tau_i\wedge T}\mathcal{L}_k^{\pi_k}F^k(s, \widehat{Z}_k^{\pi_k}(s), L(s), H(s))ds\\
&+\int_t^{\tau_i\wedge T}\sigma_k q_k^*(s)\frac{\partial F^k}{\partial\widehat{z}_k}dB_k(s) - \int_t^{\tau_i\wedge T}\omega_k\sigma_j q_j^*(s)\frac{\partial F^k}{\partial\widehat{z}_k}dB_j(s)\\
&+\int_t^{\tau_i\wedge T}(\theta_k^*(s) - \omega_k\theta_j^*(s))F_l^k\sqrt{L(s)}dW_1(s)\\
&+\int_t^{\tau_i\wedge T}\Big(F^k(s, \widehat{Z}_k^{\pi_k}(s) - (\gamma_k^*(s) - \omega_k\gamma_j^*(s))\zeta\widehat{Z}_k^{\pi_k}(s), L(s), 1)\\
&-F^k(s, \widehat{Z}_k^{\pi_k}(s), L(s), 0)\Big)dM^P(s).
\end{aligned}$$

Because the last three terms are square-integrable martingales with zero expectations, taking conditional expectations given $(t, \widehat{z}, l, h)$ on both sides of the above equation and taking (4.1) into consideration yields:

$$\begin{aligned}
&E\Big[F^k(\tau_i\wedge T, \widehat{Z}_k^{\pi_k}(\tau_i\wedge T), L(\tau_i\wedge T), H(\tau_i\wedge T))|\widehat{Z}_k^{\pi_k}(t)\\
&= \widehat{z}_k, L(t) = l, H(t) = h\Big]\\
&= F^k(t, \widehat{z}_k, l, h) + E\Big[\int_t^{\tau_i\wedge T}\mathcal{L}_k^{\pi_k}F^k(s, \widehat{Z}_k^{\pi_k}(s), L(s), H(s))ds|\widehat{Z}_k^{\pi_k}(t)\\
&= \widehat{z}_k, L(t) = l, H(t) = h\Big]\\
&\leq F^k(t, \widehat{z}_k, l, h).
\end{aligned}$$

Due to Lemma 4.1, $F^k(\tau_i\wedge T, \widehat{Z}_k^{\pi_k}(\tau_i\wedge T), L(\tau_i\wedge T), H(\tau_i\wedge T)), i = 1, 2, \ldots,$ are uniformly integrable. Thus, we have

$$\begin{aligned}
J_k(t, \widehat{z}_k, l, h) &= \sup_{\pi_k\in\Pi_k}E[\mathcal{U}_k(\widehat{Z}_k^{\pi_k}(T))|\widehat{Z}_k^{\pi_k}(t) = \widehat{z}_k, L(t) = l, H(t) = h]\\
&= \lim_{i\to\infty}E\Big[J_k(\tau_i\wedge T, \widehat{Z}_k^{\pi_k}(\tau_i\wedge T), L(\tau_i\wedge T), H(\tau_i\wedge T))|\\
&\quad\times\widehat{Z}_k^{\pi_k}(t) = \widehat{z}_k, L(t) = l, H(t) = h\Big]\\
&\leq F^k(t, \widehat{z}_k, l, h).
\end{aligned}$$

When $\pi_k = \pi_k^*$, the inequality in the above formula becomes an equality, and thus $J_k(t, \widehat{z}_k, l, h) = F^k(t, \widehat{z}_k, l, h)$. Then, the proof is complete. □

Theorem 4.3 guarantees the optimality of the solution to the HJB Eq. (4.1) and prescribes the equilibrium strategy $\pi_k^*(t), k = 1, 2$, for each insurer. However, we must check that the candidate solution $F^k$ satisfies the conditions required in Theorem 4.4. First, it is easy to show that $F^k$ is a concave solution to HJB Eq. (4.1). Therefore, we need to verify that the uniform integrability condition holds. The proof of this fact nearly follows the study performed by Zeng and Taksar (2013). More precisely, we apply the following lemma as performed by Zeng and Taksar (2013).

**Lemma 4.1.** *For $k = 1, 2$, let $\tau_i$ be the existing time from the open set $\mathcal{M}_i$, where $\mathcal{M}_i\subset\mathcal{M} = \mathbb{R}\times\mathbb{R}^+\times\{0, 1\}$ such that $\mathcal{M}_i\subset\mathcal{M}_{i+1}\subset\mathcal{M}, i = 1, 2, \ldots,$ and $\mathcal{M} = \cup_i\mathcal{M}_i$. Then, for any $\varepsilon > 1$, we have*

$$\sup_i E\Big[|J_k(\tau_i\wedge T, \widehat{Z}_k^{\pi_k^*}(\tau_i\wedge T), L(\tau_i\wedge T), H(\tau_i\wedge T))|^{\varepsilon}\Big] < \infty.$$
$$i = 1, 2, \ldots. \tag{4.31}$$

**Proof.** See the Appendix. □

**Table 4**
Model parameter values.

| Insurer 1's parameters | | Insurer 2's parameters | |
|---|---|---|---|
| Symbol | Value | Symbol | Value |
| $\mu_1$ | 0.9 | $\mu_2$ | 1 |
| $\sigma_1$ | 2 | $\sigma_2$ | 1.5 |
| $m_1$ | 1 | $m_2$ | 2 |
| $\omega_1$ | 0.2 | $\omega_2$ | 0.4 |
| Base parameters | | | |
| Symbol | Value | Symbol | Value |
| $r$ | 0.05 | $\nu$ | 1 |
| $\Lambda$ | 2 | $\alpha$ | 2 |
| $T$ | 4 | $\mathcal{K}$ | 3 |
| $\Delta$ | 0.5 | $\widehat{\rho}$ | 0.5 |
| $\zeta$ | 0.2 | $\lambda$ | 1 |

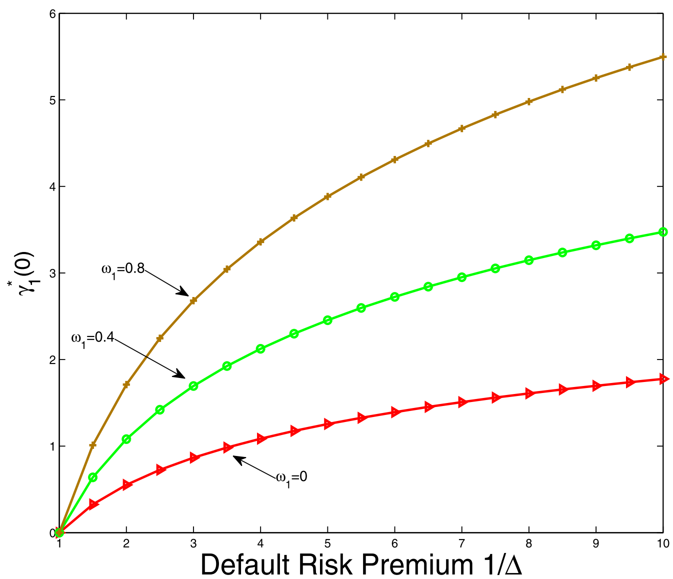


**Fig. 1.** Effect of $\frac{1}{\Delta}$ on the equilibrium strategy $\gamma_1^*(0)$.

## 5. Sensitivity analysis

To illustrate the sensitivities of the equilibrium strategies with respect to the model parameters, we conduct numerical experiments in this section. Throughout this section, we use the following model parameter values (See Table 4):

### 5.1. Equilibrium investment strategy on the corporate bond

We now analyze the sensitivity of the equilibrium strategies $\gamma_k^*(0)$ with respect to the default risk premium $\frac{1}{\Delta}$ and the loss rate $\zeta$ by varying the parameters within reasonable intervals.

Figs. 1 and 2 show a positive correlation between the equilibrium strategies $\gamma_k^*(0)$ and the default risk premium. We note that the slopes of the curves decrease as the default risk premiums increase. Figs. 3 and 4 demonstrate the negative relationship between the equilibrium strategies $\gamma_k^*(0)$ and the loss rate $\zeta$. These results are all consistent with Corollary 4.3. First, it is intuitive that an insurer would invest more wealth in a corporate bond with a higher default risk premium; it is also important that the insurer's investment in a corporate bond is zero when its risk premium is zero. Second, we note that a larger loss rate $\zeta$ produces a lower recovery amount; this implies that the potential loss of the insurer becomes larger at a higher loss rate. As in Figs. 3 and 4, insurer $k$ reduces their amount of investment in the corporate bond as the loss rate increases.

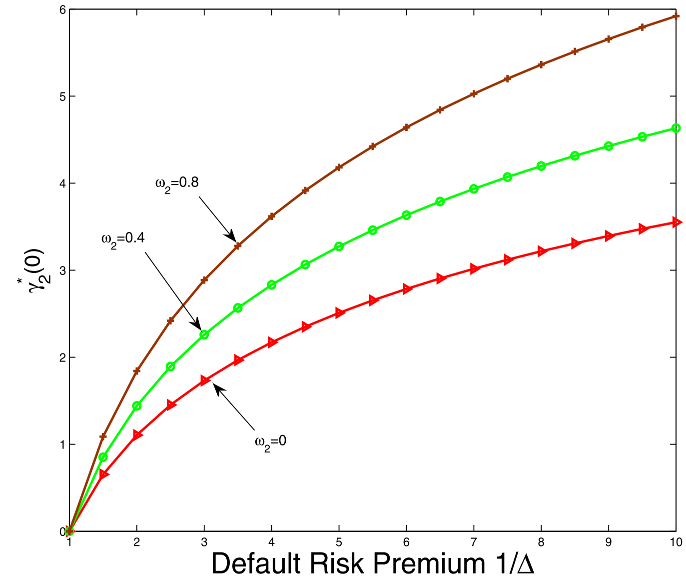


**Fig. 2.** Effect of $\frac{1}{\Delta}$ on the equilibrium strategy $\gamma_2^*(0)$.

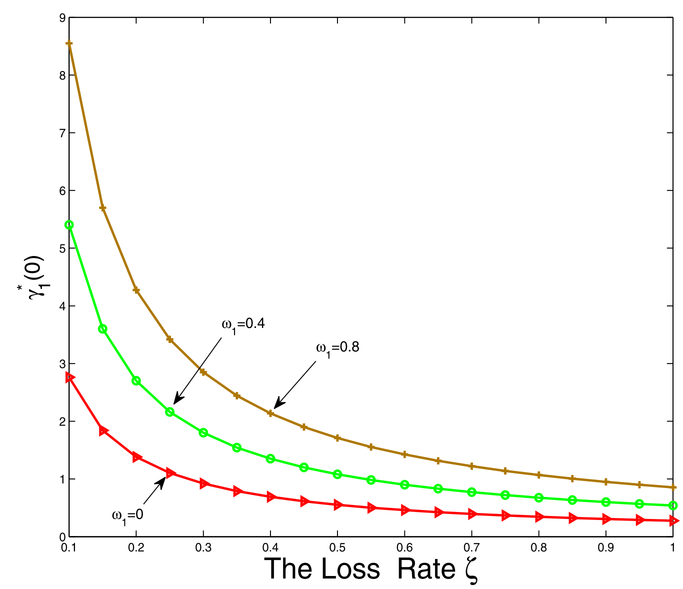


**Fig. 3.** Effect of $\zeta$ on the equilibrium strategy $\gamma_1^*(0)$.

To illustrate the effect of competition, Figs. 5 and 6 demonstrate how the equilibrium investment in the corporate bond varies with insurer $k$'s relative performance parameter $\omega_k$ for $k = 1, 2$. In Figs. 5 and 6, the equilibrium investment strategy $\gamma_k(0)$ increases as $\omega_k$ or $\omega_m$ increases. As shown by $\omega_k$ (i.e., the competition intensity faced by its competitor), a larger $\omega_k$ produces a riskier investment strategy (i.e., investing more in the corporate bond) as a response to the competition. In this case, the insurer can maximize the probability of generating greater terminal wealth against its competitor at the terminal time $T$. The effects of the competition are also illustrated in Figs. 1–4.

### 5.2. Equilibrium reinsurance strategy

Figs. 7 and 8 show that the equilibrium proportional reinsurance strategy $q_k^*(0)$ of insurer $k$ decreases as $\sigma_k$ increases when

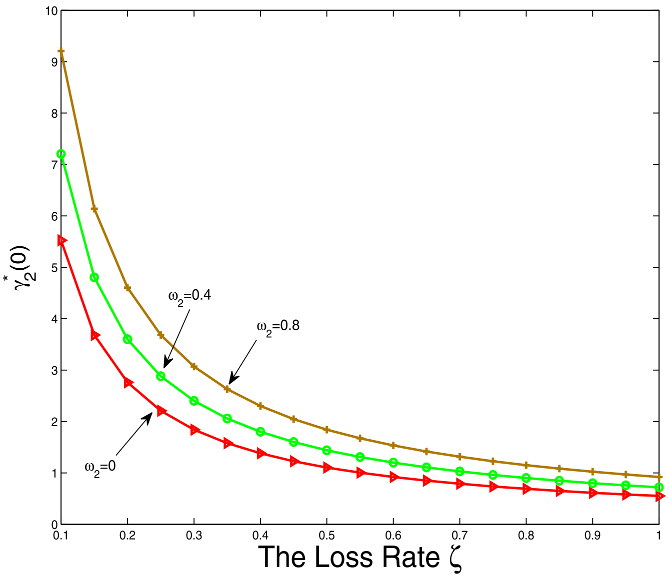


**Fig. 4.** Effect of $\zeta$ on the equilibrium strategy $\gamma_2^*(0)$.

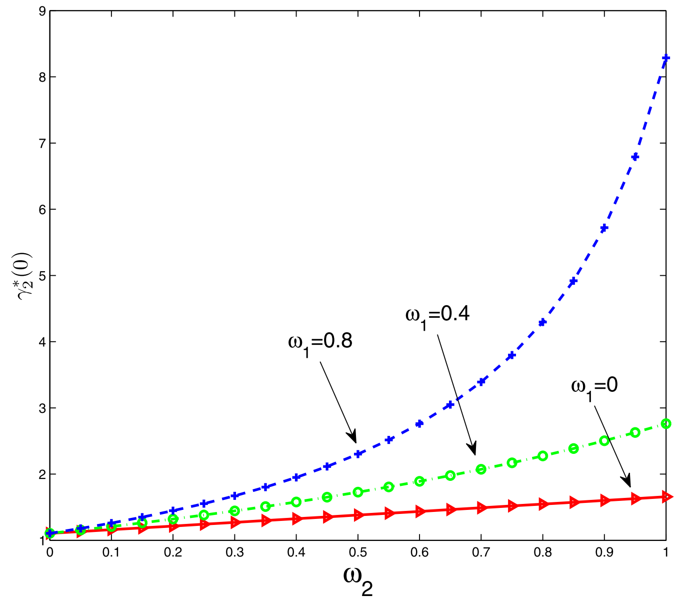


**Fig. 6.** Effect of competition on the equilibrium investment strategy $\gamma_1^*(0)$. investment strategy $\gamma_2^*(0)$.

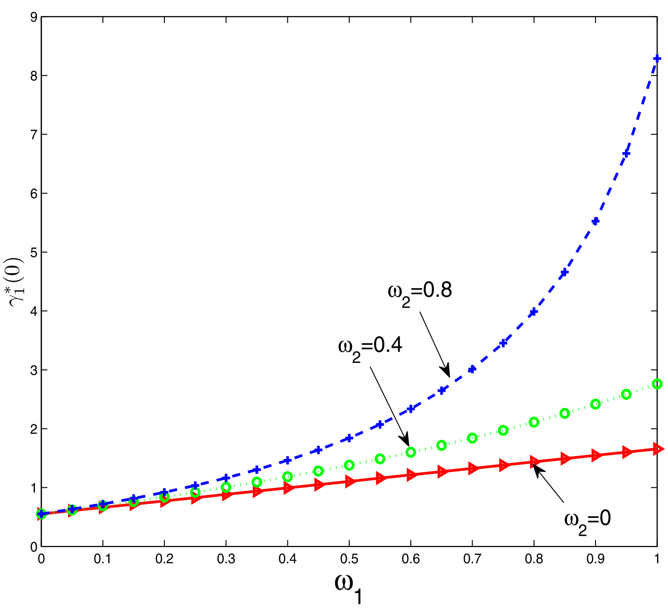


**Fig. 5.** Effect of competition on the equilibrium investment strategy $\gamma_1^*(0)$.

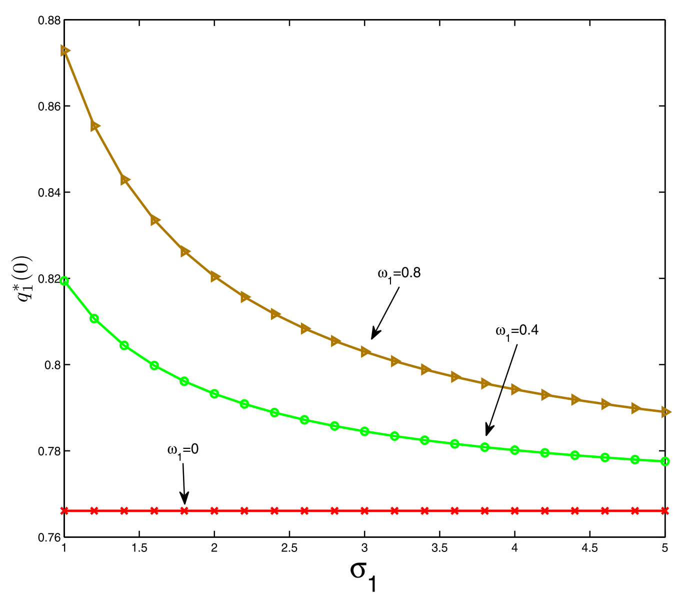


**Fig. 7.** Effect of $\sigma_1$ on the the equilibrium strategy $q_1^*(0)$.

$\omega_k > 0$; additionally, the proportional reinsurance strategy $q_k^*(0)$ is a constant when $\omega_k = 0$. These results are consistent with Theorem 4.1 and 4.2. When $\omega_k = 0$, the reinsurance strategy is consistent with that without competition in the existing literature.

From Figs. 9 and 10, we observe a positive relationship between $\omega_k$ and insurer $k$'s equilibrium proportional reinsurance strategy. Note that a larger $\omega_k$ indicates more concerns of insurer $k$ about its relative performance. Insurer $k$ may bear more risk due to the claims of purchasing fewer reinsurance contracts, while obtaining a higher terminal wealth and thus a higher probability of beating its competitor.

### *5.3. Equilibrium investment strategy*

Figs. 11–18 describe the sensitivity of the equilibrium investment strategy versus the model parameters. In Figs. 11 and 12, we analyze the effect of the correlation coefficient $\widehat{\rho}$ on the equilibrium investment strategy; and see that the equilibrium investment strategy on the stock decreases as $\widehat{\rho}$ increases; insurer $k$ tends to decrease their investment in the stock as the correlation coefficient $\widehat{\rho}$ increases. This agrees with economic intuition.

Figs. 13–16 provide graphical illustrations of the effect of $\mathcal{K}$ on the equilibrium investment strategy $\theta_k^*(0)$. We observe that the equilibrium investment strategy increases as $\mathcal{K}$ increases for $\widehat{\rho} > 0$, whereas the strategy decreases as $\mathcal{K}$ increases for $\widehat{\rho} < 0$. $\mathcal{K}$ denotes the mean reversion rate of $L(t)$ (i.e., the stock volatility); thus, a larger $\mathcal{K}$ will cause $L(t)$ to revert more quickly back to its mean $\beta$. This should lead to a more stable return of the stock. As a result, insurer $k$ will tend to increase its investment in the stock. For $\widehat{\rho} < 0$, changes in $L(t)$ and $S(t)$ move in opposite directions. If $\mathcal{K}$ increases, the hedging effect of $L(t)$ should weaken; thus, insurer $k$

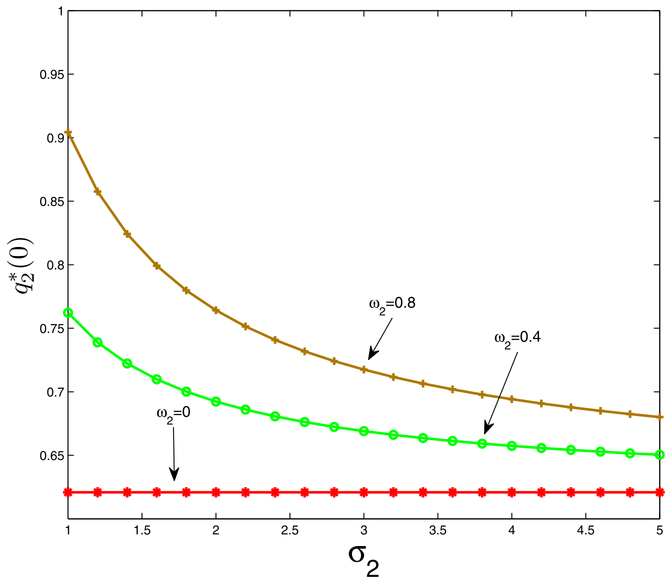


**Fig. 8.** Effect of $\sigma_2$ on the the equilibrium strategy $q_2^*(0)$.

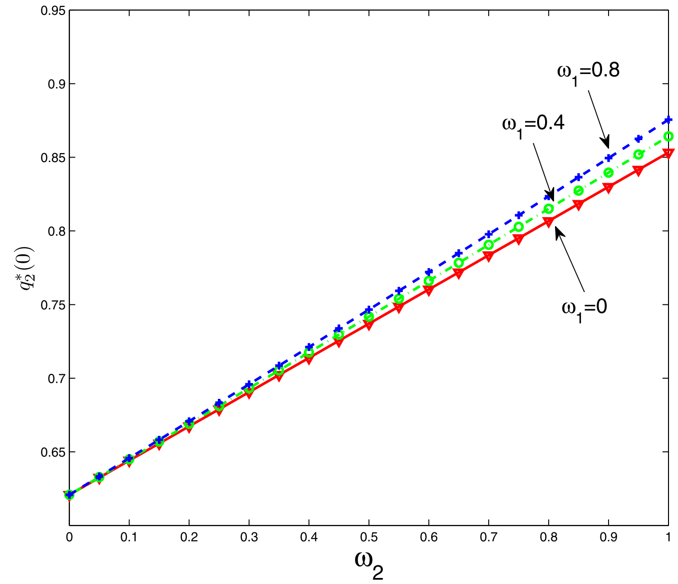


**Fig. 10.** Effect of competition on the equilibrium reinsurance strategy $q_2^*(0)$.

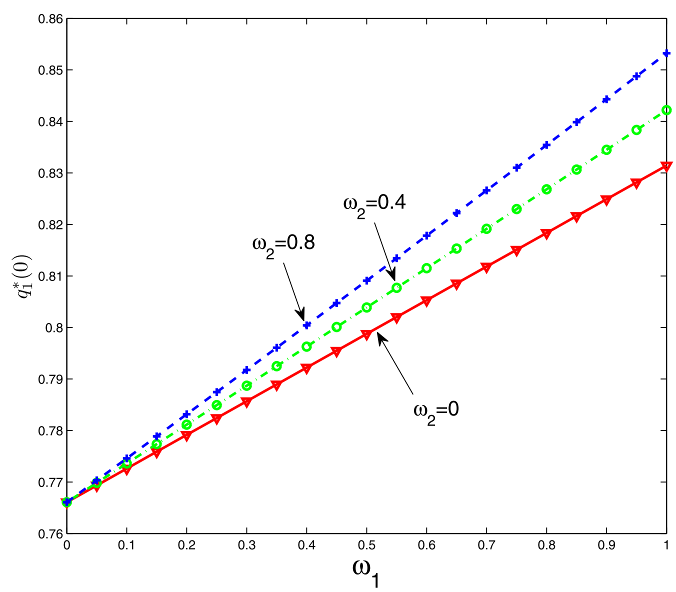


**Fig. 9.** Effect of competition on the equilibrium reinsurance strategy $q_1^*(0)$.

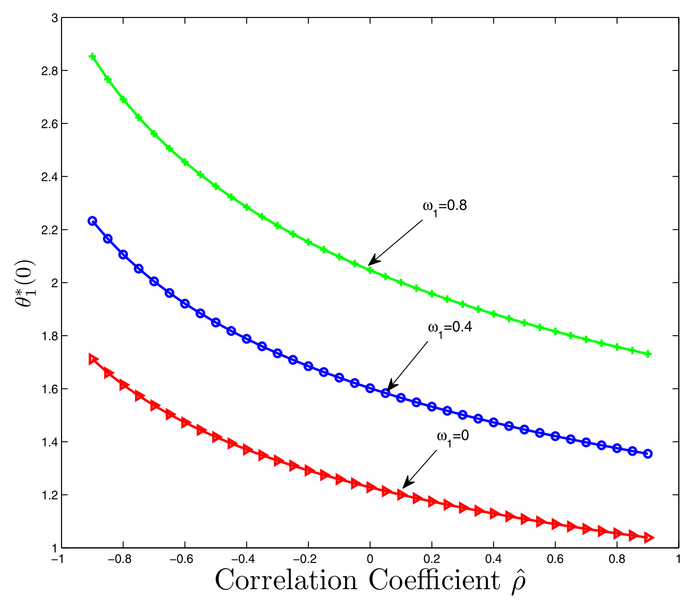


**Fig. 11.** Effect of $\widehat{\rho}$ on the the equilibrium strategy $\theta_1^*(0)$.

will reduce its investment when $\widehat{\rho} < 0$. The effect of competition is demonstrated in Figs. 17 and 18. Insurer $k$ will invest more in the stock as the dependence parameter $\omega_k$ increases; this is consistent with the analysis of the equilibrium investment strategy $\theta_k^*(t)$ in Corollary 4.2. It is optimal for each insurer to choose a riskier investment strategy $\theta_k^*(t)$ and hold more stock shares.

## 6. Conclusion

Motivated by Bensoussan et al. (2014) and Espinosa and Touzi (2015), we study stochastic differential games between two competitive insurance companies in the presence of strategic interactions driven by relative performance concerns.

We assume that the reinsurance premium is calculated by the variance premium principle and that each insurer can dynamically purchase reinsurance contracts and invest its wealth in a financial market that consists of a risk-free asset, a risky asset with stochastic volatility and a defaultable corporate bond. The goal of each insurer is to maximize its CARA utility from its terminal wealth with relative performance concerns. The optimal decision problem is modelled as a non-zero-sum game. We obtain the Nash equilibrium strategies for the game by solving the corresponding HJB equations. We find that relative concerns distort insurers' rational decisions, in that each insurer decreases their purchase of reinsurance contracts and holds more risky assets (i.e., stock and defaultable corporate bond) in the presence of competition. Finally, we establish a verification theorem for the optimality of the given control policies. Numerical examples are provided to illustrate the effects of the model parameters and the relative concerns on the equilibrium strategies.

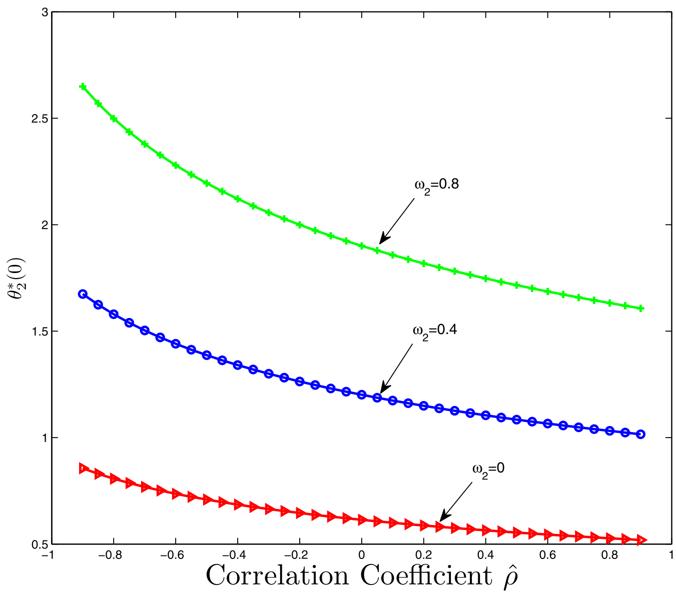


**Fig. 12.** Effect of $\hat{\rho}$ on the the equilibrium strategy $\theta_2^*(0)$.

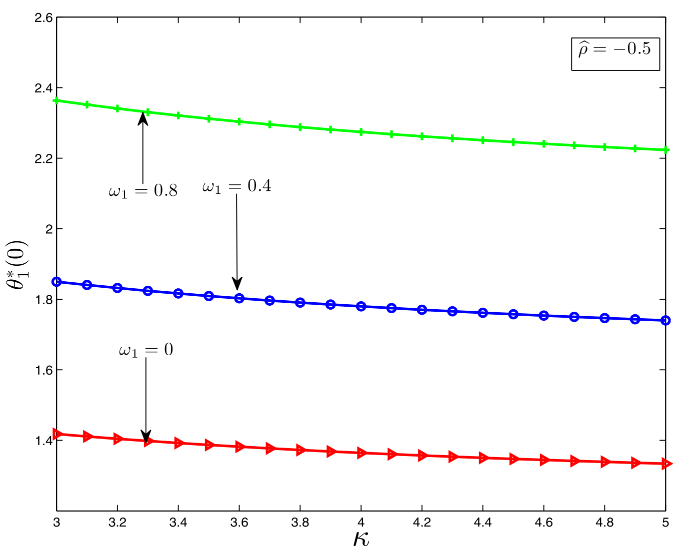


**Fig. 13.** Effect of $\mathcal{K}$ on the the equilibrium strategy $\theta_1^*(0)$, when $\hat{\rho} = -0.5$.

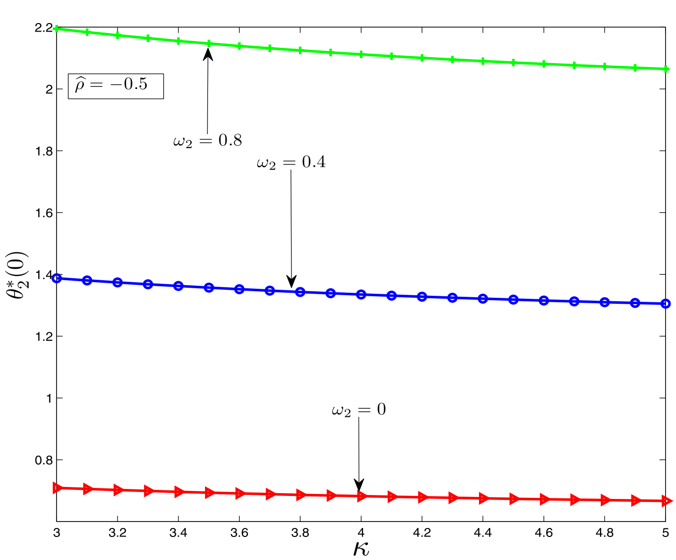


**Fig. 14.** Effect of $\mathcal{K}$ on the the equilibrium strategy $\theta_2^*(0)$, when $\hat{\rho} = -0.5$.

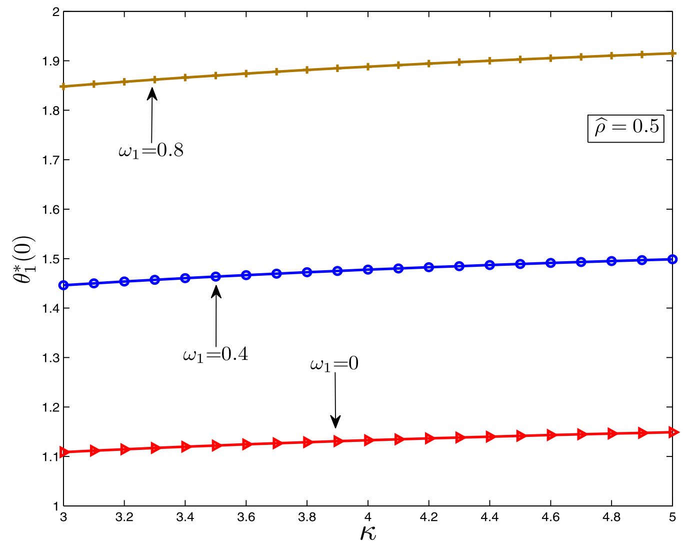


**Fig. 15.** Effect of $\mathcal{K}$ on the the equilibrium strategy $\theta_1^*(0)$, when $\hat{\rho} = 0.5$.

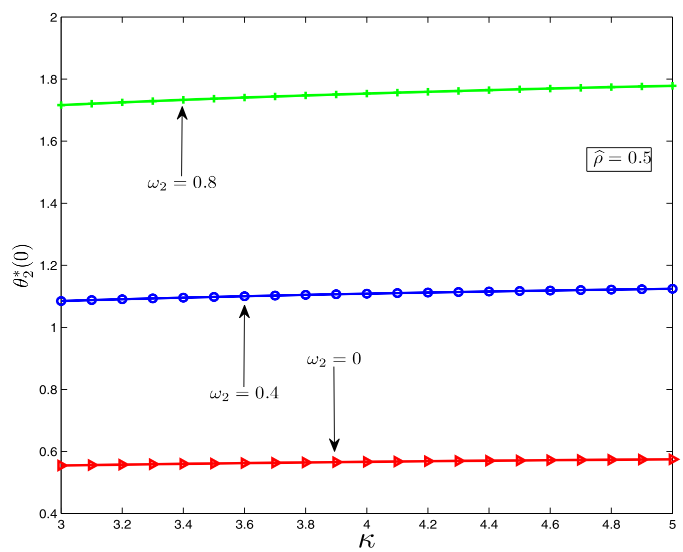


**Fig. 16.** Effect of $\mathcal{K}$ on the the equilibrium strategy $\theta_2^*(0)$, when $\hat{\rho} = 0.5$.

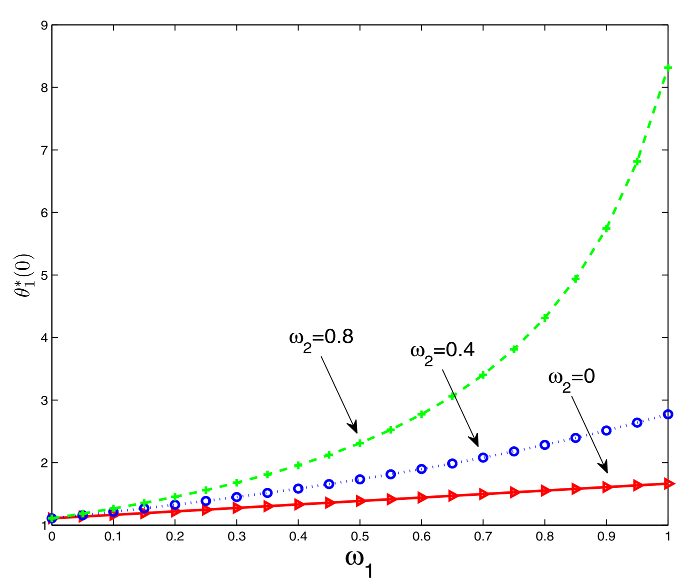


**Fig. 17.** Effect of competition on the equilibrium investment strategy $\theta_1^*(0)$.

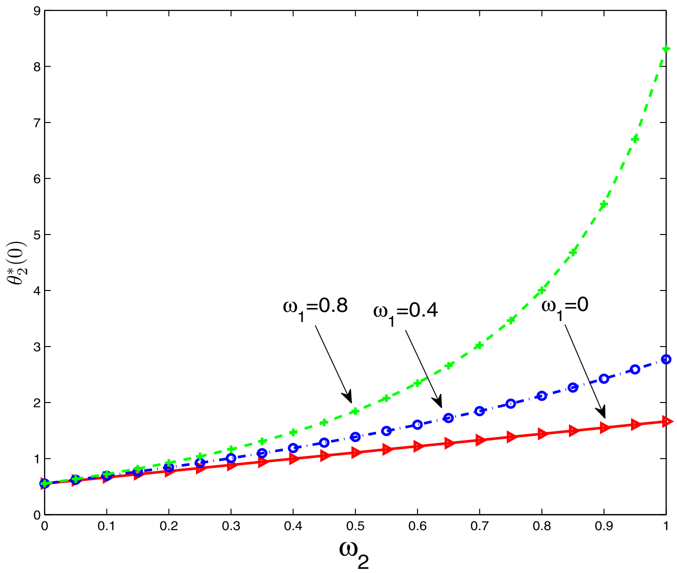


**Fig. 18.** Effect of competition on the equilibrium investment strategy $\theta_2^*(0)$.


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## Appendix A. Proof of Lemma 4.1

**Proof.** Note that

$$J_k(t,\widehat{Z}_k^{\pi_k^*}(t),L(t),H(t)) = (1-H(t))J_k(t,\widehat{Z}_k^{\pi_k^*}(t),L(t),0) + H(t)J_k(t,\widehat{Z}_k^{\pi_k^*}(t),L(t),1).$$

For $0\le H(t)\le 1$, we only need to verify that $\sup_i E[|J_k(\tau_i\wedge T,\widehat{Z}_k^{\pi_k^*}(\tau_i\wedge T),L(\tau_i\wedge T),j)|^{\varepsilon}]<\infty, j=0,1.$

(1) Case of $H(t)=0$. Using Eq. (3.6), we obtain

$$\begin{aligned}
J_k(t,\widehat{Z}_k^{\pi_k^*}(t),L(t),0) &= -\frac{1}{m_k}\exp\{-m_k\widehat{Z}_k^{\pi_k^*}(t)e^{r(T-t)}+g_{0k}(t)\\
&+f_{0k}(t)L(t)\} = -\frac{1}{m_k}\exp\Big\{-m_k\widehat{z}_k e^{r(T}-m_k\int_0^t e^{r(T-s)}\\
&\times\Big[(d_k-\omega_k d_j)-(\delta(q_k^*(s))-\omega_k\delta(q_j^*(s)))-(q_k^*(s)n_k-\omega_k q_j^*(s)n_j)\\
&+(\theta_k^*(s)-\omega_k\theta_j^*(s))\alpha L(s)+(\gamma_k^*(s)-\omega_k\gamma_j^*(s))\eta(1-\Delta)\Big]ds\\
&-m_k\widehat{\rho}\int_0^t e^{r(T-s)}(\theta_k^*(s)-\omega_k\theta_j^*(s))\sqrt{L(s)}dW_2(s)\\
&-m_k\int_0^t e^{r(T-s)}\sqrt{1-\widehat{\rho}^2}(\theta_k^*(s)-\omega_k\theta_j^*(s))\sqrt{L(s)}d\overline{W}(s)\\
&-\frac{m_k^2(1-\widehat{\rho}^2)}{2}\int_0^t e^{2r(T-s)}(\theta_k^*(s)-\omega_k\theta_j^*(s))^2L(s)ds\\
&+\frac{m_k^2(1-\widehat{\rho}^2)}{2}\int_0^t e^{2r(T-s)}(\theta_k^*(s)-\omega_k\theta_j^*(s))^2L(s)ds\\
&+m_k\int_0^t e^{r(T-s)}(\gamma_k^*(s)-\omega_k\gamma_j^*(s))\zeta dM^P(s)\\
&-m_k\int_0^t e^{r(T-s)}\sigma_k q_k^*(s)dB_k(s)+m_k\int_0^t e^{r(T-s)}\omega_k\sigma_j q_j^*(s)dB_j(s)\\
&+g_{0k}(t)+f_{0k}(t)L(t)\Big\}
\end{aligned}$$

Here, we assume that

$$W_1(t)=\widehat{\rho}W_2(t)+\sqrt{1-\widehat{\rho}^2}\,\overline{W}(t),$$

where $\overline{W}(t)$ is a standard Brownian motion independent of $W_2(t)$. Let

$$\phi_{1k}(t) = -m_k\alpha(\theta_k^*(t)-\omega_k\theta_j^*(t))e^{r(T-t)} + \frac{m_k^2(1-\widehat{\rho}^2)}{2}(\theta_k^*(t)-\omega_k\theta_j^*(t))^2e^{2r(T-t)},$$

$$\phi_{2k}(t) = -m_k\widehat{\rho}(\theta_k^*(t)-\omega_k\theta_j^*(t))e^{r(T-t)}, \quad \phi_{3k}(t)=f_{0k}(t).$$

From Corollary 4.1, we have $f_{0k}(t)$, which satisfies the following ODE

$$f'_{0k}(t)+\frac{1}{2}\nu^2(1-\widehat{\rho}^2)f_{0k}^2(t)-(\widehat{\rho}\alpha\nu+\mathcal{K})f_{0k}(t)-\frac{\alpha^2}{2}=0.$$

Direct calculation yields

$$\phi_{1k}+\phi'_{3k}-\mathcal{K}\phi_{3k}+\frac{1}{2}(\phi_{3k}\nu+\phi_{2k})^2=0$$

Because $\nu,\alpha,m_k,f_{0k}(t),\pi_k^*(t),k=1,2,$ are bounded, there exist constants $\Theta$, $\Gamma>0$, s.t.

$$\frac{\varepsilon^2-\varepsilon}{2}\Big[e^{2r(T-t)}m_k^2\sigma^2(1-\widehat{\rho}^2)(\theta_k^*)^2+(\phi_{3k}\sigma+\phi_{2k})^2\Big]<\Theta,$$

and

$$\widehat{\rho}\alpha\sigma\varepsilon+(\widehat{\rho}^2\nu\sigma+\sigma^2)f_{0k}(t)\varepsilon<\Gamma.$$

Let

$$\xi_k^+=\frac{-\Gamma+\sqrt{\Gamma^2+2(\Theta+1)\varepsilon}}{2},$$

$$\xi_k^-=\frac{-\Gamma-\sqrt{\Gamma^2+2(\Theta+1)\varepsilon}}{2},$$

$$a(t)=\frac{\xi_k^+e^{\xi_k^+t}-\xi_k^-e^{\xi_k^-t}}{e^{\xi_k^+t}-e^{\xi_k^-t}}.$$

Then $a(t)>0$, and $a'(t)+\Gamma a(t)+\frac{a(t)^2}{2}=-(\Theta+1)$ holds.

Define

$$\Phi_{1k}=\varepsilon\phi_{1k}+e^{2r(T-t)}\frac{\varepsilon^2-\varepsilon}{2}m_k^2\sigma^2(1-\widehat{\rho}^2)(\theta_k^*)^2, \quad \Phi_{2k}=\varepsilon\phi_{2k},$$

$$\Phi_{3k}=\varepsilon\phi_{3k}+\varepsilon a(t).$$

Consequently, we have

$$\begin{aligned}
&|J_k(\tau_i\wedge T,Z^{\pi_k^*}(\tau_i\wedge T),L(\tau_i\wedge T),0)|^{\varepsilon}\\
&\le m_k^{-\varepsilon}\exp\Big\{-m_k\varepsilon\widehat{z}_ke^{rT}-m_k\int_0^t e^{r(T-s)}\varepsilon\Big[(d_k-\omega_kd_j)-(\delta(q_k^*(s))\\
&-\omega_k\delta(q_j^*(s)))-(q_k^*(s)n_k-\omega_kq_j^*(s)n_j)\\
&+(\gamma_k^*(s)-\omega_k\gamma_j^*(s))\eta(1-\Delta)\Big]ds-m_k\int_0^t e^{r(T-s)}\varepsilon\sigma_kq_k^*(s)dB_k(s)\\
&+m_k\int_0^t e^{r(T-s)}\varepsilon\omega_k\sigma_jq_j^*(s)dB_j(s)+\varepsilon g_{0k}(s)\\
&+m_k\int_0^t e^{r(T-s)}\varepsilon(\gamma_k^*(s)-\omega_k\gamma_j^*(s))\zeta dM^P(s)\Big\}\\
&\times\exp\Big\{-m_k\int_0^t e^{r(T-s)}\varepsilon\sqrt{1-\widehat{\rho}^2}(\theta_k^*(s)-\omega_k\theta_j^*(s))\sqrt{L(s)}d\overline{W}(s)\\
&-\frac{m_k^2(1-\widehat{\rho}^2)}{2}\varepsilon^2\int_0^t e^{2r(T-s)}(\theta_k^*(s)-\omega_k\theta_j^*(s))^2L(s)ds\Big\}
\end{aligned}$$

$$\times\exp\left\{\int_0^t\Phi_{1k}(s)L(s)ds+\int_0^t\Phi_{2k}(s)\sqrt{L(s)}dW_2(s)+\Phi_{3k}(t)L(t)\right\}.$$

By Lemma A.2 of Zeng and Taksar (2013), we obtain

$$\begin{aligned}&\Phi_{1k}+\Phi_{3k}'-\Phi_{3k}\mathcal{K}+\frac{(\Phi_{3k}\sigma+\Phi_{2k})^2}{2}\\&=\varepsilon\left(a'-a\mathcal{K}+(\widehat{\rho}\alpha\sigma\varepsilon+(\widehat{\rho}^2\nu\sigma+\sigma^2)f_{0k}(t)\varepsilon)a+\frac{\varepsilon\sigma^2a^2}{2}\right)\\&\quad+\frac{\varepsilon^2-\varepsilon}{2}\left[e^{2r(T-t)}m_k^2\sigma^2(1-\widehat{\rho}^2)(\theta_k^*)^2+(\phi_{3k}\sigma+\phi_{2k})^2\right]\\&\leq\epsilon\left(a'+a\Gamma+\frac{\varepsilon\sigma^2a^2}{2}+\Theta\right)=\varepsilon(-\Theta-1+\Theta)=-\varepsilon<0\end{aligned}$$

Then, according to A.1 of Zeng and Taksar (2013), we have

$$\begin{aligned}&E\left[\exp\left\{\int_0^t\Phi_{1k}(s)L(s)ds+\int_0^t\Phi_{2k}(s)\sqrt{L(s)}dW_2(s)+\Phi_{3k}(t)L(t)\right]\\&\leq\exp\left(\mathcal{K}\beta\Phi_{3k}(t)\right)\end{aligned}\tag{A.1}$$

Using the Novikov condition,

$$\begin{aligned}&\exp\left\{-m_k\int_0^t e^{r(T-s)}\varepsilon\sqrt{1-\widehat{\rho}^2}(\theta_k^*(s)-\omega_k\theta_j^*(s))\sqrt{L(s)}d\overline{W}(s)\right.\\&\left.-\frac{m_k^2(1-\widehat{\rho}^2)}{2}\varepsilon^2\int_0^t e^{2r(T-s)}(\theta_k^*(s)-\omega_k\theta_j^*(s))^2L(s)ds\right\}\end{aligned}$$

can be considered to be a local martingale. In addition, because $0\leq H(t)\leq 1$, we have:

$$\begin{aligned}&E\left[\exp\left\{-m_k\varepsilon\int_0^t e^{r(T-s)}(\gamma_k^*(s)-\omega_k\gamma_j^*(s))\eta(1-\Delta)ds\right.\right.\\&\left.\left.+m_k\varepsilon\int_0^t e^{r(T-s)}(\gamma_k^*(s)-\omega_k\gamma_j^*(s))\zeta dM^p(s)\right\}\right]<\infty.\end{aligned}$$

Therefore,

$$E\left[|J_k(\tau_i\wedge T,\widehat{Z}_k^{\pi_k^*}(\tau_i\wedge T),L(\tau_i\wedge T),0)|^{\varepsilon}\right]<\infty,\quad i=1,2,\ldots.\tag{A.2}$$

(2) Case of $H(t)=1$. Using a similar method, we have

$$E\left[|J_k(\tau_i\wedge T,\widehat{Z}_k^{\pi_k^*}(\tau_i\wedge T),L(\tau_i\wedge T),1)|^{\varepsilon}\right]<\infty,\quad i=1,2,\ldots.\tag{A.3}$$

Combining (A.2) with (A.3), we obtain

$$E\left[|J_k(\tau_i\wedge T,\widehat{Z}_k^{\pi_k^*}(\tau_i\wedge T),L(\tau_i\wedge T),H(\tau_i\wedge T))|^{\varepsilon}\right]<\infty,\quad i=1,2,\ldots.\tag{A.4}$$

□

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