Insurance: Mathematics and Economics 79 (2018) 210–224

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Optimal investment management for a defined contribution pension fund under imperfect information☆

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HIGHLIGHTS

- There exist both the observable and unobservable states in the financial market.
- The dynamics of the unobservable state is described by a Hidden Markov chain.
- The growth rate of assets and salary are modulated by the financial market states.
- The optimal investment problem of DC pension fund is solved explicitly.
- Numerical results are present to show the impact of the imperfect information.

ARTICLE INFO

Article history:
Received February 2017
Received in revised form January 2018
Accepted 23 January 2018
Available online 1 February 2018

Keywords:
Imperfect information
Hidden Markov model
Stochastic salary
Defined contribution pension fund
Sufficient statistics

ABSTRACT

This paper investigates an optimal multi-period investment management problem for a defined contribution pension fund under the mean–variance criterion with imperfect information, meaning that both observable and unobservable states exist in the financial market. The dynamics of the unobservable market state process are formulated by a discrete-time finite-state hidden Markov chain with time-varying transition probability matrices. Due to the long investment horizon of a defined contribution pension fund, our paper considers only risky assets whose returns depend on both the observable and unobservable market states. Meanwhile, the stochastic salary process is also modulated by the observable and unobservable market states. By adopting sufficient statistics, the portfolio optimization problem for the defined contribution pension fund with imperfect information is transformed into one with complete information. Then, the optimal investment strategy and the efficient frontier are explicitly derived using the dynamic programming approach and the Lagrange dual method. Finally, numerical results show that the imperfection of market state information may cause a loss of investment return.

1. Introduction

Recent decades have witnessed the widespread use of defined contribution (DC) pension funds all over the world due to the aging population and the longevity risk. Compared with the defined benefit (DB) pension fund, the contribution rate of DC pension funds is usually preset, and the benefit depends on the investment return in the financial market during the accumulation period before retiring. The DC pension fund thus has an advantage over the DB pension fund by transferring the investment risk to the retiree from the pension fund sponsor. This is why more and more countries are beginning to partially or fully shift their pension fund system from the DB to the DC scheme. Corresponding to this trend, research on DC pension fund asset management has become a hot topic in the fields of finance and actuarial science. In the past two decades, the optimal investment management problem for DC pension funds has been extensively studied under the CRRA or CARA framework by maximizing the expected utility of the terminal wealth; see, for example, Boulier et al. (2001), Haberman and Vigna (2002), Cairns et al. (2006), Gao (2008), Di Giacinto and Vigna (2012), and Blake et al. (2013). In addition, based on the development of the dynamic mean–variance portfolio selection theory, more studies are emerging on the optimal DC pension fund

☆ This research is supported by the National Natural Science Foundation of China (Nos. 71601055, 71603061, 71471045, 71671047, 71501050), the Philosophy and Social Science Programming Foundation of Guangzhou (No: 15G40), the Innovation Team Project of Guangdong Colleges and Universities (No. 2016WCXTD012), the Foundation for Young Scholars of Guangzhou (No: 17QNXR07), the Natural Science Foundation of Guangdong Province of China (Nos. 2017A030313399, 2015A030313574, 2017A030313397), the Innovative School Project in Higher Education of Guangdong Province of China (No. GWTP-GC-2017-03), The Humanities and Social Science Research Foundation of the National Ministry of Education of China (No. 15YJAZH051), and the Fund Project of Guangdong University of Foreign Studies (Nos. 16QN14, TD1604). Guangdong High-level University Project: Study on the Construction of an Innovation Mechanism of Integrating Economics and Management Majors-In Service of the Strategic Need of Building the 21st-Century Maritime Silk Road.

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https://doi.org/10.1016/j.insmatheco.2018.01.007
0167-6687/

management problem under the mean–variance criterion, such as Højgaard and Vigna (2007), He and Liang (2013), Yao et al. (2013, 2014, 2016a), Vigna (2014), and Wu et al. (2015). As the investment time horizon for a DC pension fund is usually quite long, most studies focus on the effect of the various risks of a DC pension fund investment strategy, such as the inflation risk, stochastic salary risk, investment risk, and so on in a continuously trading financial market. However, mean and variance, the two indexes used to measure return and risk in an investment portfolio, can easily be calculated from observed data on the financial market and are frequently used in financial asset management. Therefore, it is of great importance to further investigate the multi-period investment management problem for a DC pension fund under the mean–variance criterion. In this paper, we therefore try to solve the optimal mean–variance investment problem for a DC pension fund in a discrete-time financial market.

Since the pioneering work of Markowitz (1952), numerous studies have emerged on the optimal mean–variance investment problem. In particular, the successful dynamic mean–variance portfolio selection proposed by Li and Ng (2000) and Zhou and Li (2000) has stimulated enthusiasm among scholars to investigate dynamic mean–variance optimization investment problems. Although there has been some research on the mean–variance DC pension fund management problem, further exploration is needed. To make the optimization problem more tractable, studies commonly assume that the decision-maker can obtain all of the financial market information when making an investment decision, which is known as the complete information assumption. However, this assumption contradicts the investment reality. Usually, at the moment of decision-making, the investment decision can only be made based on known information rather than the entire body of information in the financial market. This implies that the decision-making process depends on imperfect information. Particularly for a DC pension fund with a long-term investment horizon, the decision-maker's judgement must be updated with market information over time. This makes it is essential to consider the effect of imperfect information on an investment strategy for a DC pension fund. Thus, in this paper, we explore an optimal multi-period investment problem for a DC pension fund in a financial market with both observable and unobservable market information, which extends the current research on the multi-period DC pension fund to include the case with imperfect information.

In a setting with completely observable information, the model parameters are assumed to be constant or to be deterministic functions of time, making the investment optimization problem more tractable using the optimal control theory and method. In practice, however, the performance of the financial market depends on a range of economic, financial, policy, natural and political factors. Some information, such as interest rates, stock prices, exchange rates, and so on, can be directly observed in various ways. Yet there are also many unobservable factors, for example, the bull/bear market state. Although these factors cannot be directly observed and used to make investment decisions, they affect the evolution of the financial market. Even in the security market with its perfect transparency, most investors cannot access a lot of market information, such as the high frequency transition data of stocks. The studies of Fama (1965), Keown and Pinkerton (1981), and Fama and French (1992) show that the security market in the USA is weak-form efficient, which means that financial market information can only be partially known by investors. Moreover, due to a variety of limitations, decision-makers may not try their best to consider all potentially related economic variables and market states to forecast the future returns on the financial assets, but rather select a few important states, for example, bull/bear market states. Thus generally, decision-makers can only make their investment decision based on imperfect information rather than the complete market information.

On the other hand, the performance of model parameters (such as the return on financial assets) is not independent of the macroeconomic environment. Many economic variables (such as changes in political/economic policies, innovations in technique, bull/bear markets, wars, natural disasters, and the growth of GDP and CPI) have significant impact on the return rates of financial assets. At the moment of decision-making, investors have access to only a few of these variables (such as a bull/bear market), while many other variables (such as innovations in technique) cannot be known, although they also affect the return rate of the financial assets and some of this information is contained in the investors' observations. Hence, investors usually try to gather as much information as possible, so that they can infer the expected return on the financial assets based on their observations of the financial market. Thereafter, investors update their investment decisions in light of newly observed information. When there is a long history of observation data, estimations of the model parameters inevitably fluctuate according to the shifts in these market variables.

The hidden Markov model (HMM) is often used to describe the dynamics of this kind of imperfectly observable information process. In recent decades, HMM has been extensively used to investigate the portfolio optimization problem; for examples see Honda (2003), Bäuerle and Rieder (2005), Bensoussan et al. (2009), Elliott et al. (2010), Çanakoğlu and Özekici (2011), Yao and Li (2013), and Bae et al. (2013). The above mentioned studies on the portfolio optimization problem under HMM show that the imperfectly observable information noticeably influences the optimal investment strategy, the investment benefit, and the beliefs of decision-makers. In particular, the result of this imperfect information is that the optimal investment on a risky asset depends on the investment time horizon, and the hedging demand of an optimal investment strategy depends significantly on the estimation of the model parameters, which can no longer be ignored. However, until now, little research has focused on the effect of imperfect information on the investment strategy for DC pension funds, which have a relatively long investment horizon. During this lengthy investment period, both the observable and unobservable market states inevitably change over time. Although decision-makers cannot receive all of the information about all of these financial market states, additional information continually comes to light and the estimation of model parameters is updated. Undoubtedly, the imperfect information about the financial market should be taken into account during the investment process of the DC pension fund. This is a major motivation of this paper.

In this paper, we consider an optimal multi-period investment problem for a DC pension fund where the financial market has both unobservable and observable states. The dynamics of the unobservable market state are described by a discrete-time finite-state hidden Markov chain, and the transition matrices are assumed to be time-varying rather than constant. The financial market consists of multiple risky assets whose return rates depend on both the observable and unobservable market states at that period. At each time period, decision-makers receive a stochastic salary whose growth rate depends on both the observable and unobservable market states, and they contribute a certain amount of their wealth to the pension fund based on the observed information about the financial market. Under the mean–variance criterion, the decision-makers allocate their wealth among these risky assets on the basis of the observed market state information up to the current moment. This is a discrete-time optimization problem under imperfect information. By adopting the sufficient statistics method, the optimal mean–variance DC pension fund management problem under imperfect information is transformed into one with complete information. The main contribution of this paper is to model and solve an optimal multi-period mean–variance DC pension fund management problem in a market with imperfectly observable information. Moreover, the expressions of the optimal strategy and the efficient frontier are explicitly derived. The other contribution of this paper is to analyze and demonstrate the effect of the imperfect information on an efficient investment strategy and the efficient frontier of the DC pension fund management problem.

The main findings for the optimal investment strategy and the investment return of the dynamic mean–variance DC pension fund in this paper are given as follows.

(i) Compared with the multi-period portfolio optimization model of DC pension fund with complete information, the imperfect information reduces the investment return for the same level of investment risk, and this result coincides with the continuous-time portfolio optimization model (Xia, 2001). In other words, the imperfectly observed information indeed causes the investment loss, so that the recognition and application of imperfect information model are economically valuable. In particular, when there is more positive information received, the optimal investment amount on risky assets under the complete information model is greater than that under the hidden Markov model. However, this phenomenon inverts when there are more negative information observed.

(ii) For the same level of expected terminal wealth, the investment risk, which is measured by the variance of the terminal wealth, clearly reduces with the decrease in the volatility of salary's growth rate. As we can expect, given a lower volatility of salary's growth rate, the contribution amount at each time period to the pension fund account becomes less uncertain. Then, in the case of the same conditions for the market environment, this kind of liquidity risk decreases at each time period. Therefore, the aggressive investment risk over the whole investment horizon decreases.

(iii) When close to the end of investment, the optimal investment amount on risky asset reduces, which is consistent with the investment practice in the financial market. Generally, the financial advisors always suggest the clients to reduce their investment amount on risky asset when close to the retirement that is referred to as the "age effect" (Campbell, 2006).

The remainder of this paper is organized as follows. Section 2 describes the optimization model of mean–variance DC pension fund management with imperfect information in a discrete-time financial market setting. In Section 3, using the sufficient statistics method, the mean–variance portfolio optimization problem with imperfect information is converted into an optimization problem with complete information. Section 4 explicitly derives the optimal investment strategy, the value function, and the efficient frontier of the mean–variance DC pension fund management problem. Some special cases are given in Section 5. Section 6 presents numerical results for the effect of imperfect information on the efficient investment strategy and the efficient frontier. Finally, Section 7 concludes the paper.

2. Model formulation

Suppose that the decision-maker of a DC pension fund begins to allocate her/his wealth in the financial market at time 0 equipped with the initial wealth $W_0 > 0$, and wants to conduct a multi-period investment activity over the planned time horizon T. The investment horizon is divided into T periods, where the tth period represents the time interval $[t-1, t)$ for $t = 1, 2, \dots, T$. The decision-maker can allocate her/his wealth among multiple risky assets at the beginning of each period without paying any transaction fees. Assume that both observable and unobservable states exist in the financial market, and that the decision-maker can only obtain information on the observable market state. As time passes, the decision-maker continues to receive information to update her/his judgements about the returns on the financial assets.

2.1. Hidden Markov model

Let U_t denote the unobservable market state at time t, and assume that the process of the unobservable market state, $U = \{U_t; t = 0, 1, \dots, T\}$, is a discrete-time Markov chain with a finite-state space $F = \{1, \dots, i, \dots, n\}$ and a time-dependent transition matrix $Q_t = \left(q_t(i,l)\right)_{n\times n}$, where

$$q_t(i,l) = \Pr\left\{U_{t+1} = l | U_t = i\right\}$$

is the transition probability of the unobservable market state from $U_t = i$ at time t to $U_{t+1} = l$ at time $t+1$ for $t = 0, 1, \dots, T-1$. Notice that U_t is unobservable, so U is a finite-state hidden Markov chain.

Associated with the unobservable market state U_t is a random variable O_t, which is the observable market state at time t and takes values in a finite-state space $S = \{1, \dots, j, \dots, m\}$. By observing O_t at time t, information regarding the true unobservable market state U_t is obtained by the decision-maker. Suppose that the performance of the financial market (the profits of risky assets) evolves according to both U and O, while the decision-maker can only observe the information process O. We assume that O_t is the only reflection of the unobservable state at that time point, U_t, which implies that O_t is independent of all the past history information $O_l(l < t)$ and $U_l(l < t)$. In other words,

$$\Pr\{O_t = j | U_t, U_{t-1}, \dots, U_0; O_{t-1}, \dots, O_0\} = \Pr\{O_t = j | U_t\}.$$

At time t, when $U_t = i$, we assume that an observation will have message $O_t = j$ with probability $\delta_t(i,j)$, i.e. $\delta_t(i,j) = \Pr\{O_t = j | U_t = i\}$. Define the information matrix as $\Delta_t = (\delta_t(i,j))_{n\times m}$, $i \in F, j \in S$.

We denote the whole accumulatively observed information up to time t by $\overline{O}_t = (\dots, O_{-1}, O_0, O_1, \dots, O_t)$ and $I_t = \{U_t, \overline{O}_t\}$. Let $\Im_t^O$ denote the σ-field generated by the observed information up to time t, i.e. $\Im_t^O = \sigma\{(\dots, O_{-1}, O_0, O_1, \dots, O_t)\}$ for $t = 0, 1, \dots, T-1$.

2.2. Wealth process and portfolio optimization problem under imperfect information

To maintain her/his living standard after retirement, the decision-maker of the DC pension fund needs to convert the pension fund into an annuity that delivers a programmed pension at each period after retiring. Before retirement, the decision-maker has to contribute a certain amount of money to the pension fund account at each period in a predefined way. Thus, salary is the first key point the decision-maker should consider. Let s_t be the salary the decision-maker receives at time t. In this paper, we assume that, at each time period, the decision-maker's salary level is stochastic and its dynamic process is described as

$$s_{t+1} = \nu_t(U_t, O_t)s_t, \quad t = 0, 1, \dots, T-1, \tag{1}$$

where s_0 is the initial salary. Notice that, in practice, the salary is usually random and its growth rate fluctuates according to macroeconomic states, which may not be completely known (observed) by the decision-maker. Thus, we suppose that $\nu_t(U_t, O_t)$, the random salary growth rate over period $t+1$, is determined by both the observable and unobservable market state. In the long run, the salary level mostly tends to increase. So, without loss of generality, we assume that $\nu_t(U_t, O_t) > 0$ for almost surely all $t = 0, 1, \dots, T-1$ given the unobservable market state U_t and the observable market state O_t. At the beginning of each time period, based on the market state observed at that moment, O_t, the decision-maker needs to determine how much of the salary, $c_t(O_t)s_t$, to contribute to the DC pension fund. Here $c_t(O_t)$ is a deterministic contribution rate based on the observed information at time t.

Due to the long investment horizon for a DC pension fund, usually 30–40 years, almost all of the return rates on the financial assets will fluctuate according to the financial market state over time. Thus, here we consider a financial market with $L+1$ risky assets with random return rates. Denote the return rates of the risky assets at time period $t+1$ within the planning horizon by the vector $\mathbf{r}_t(U_t, O_t) = [r_t^0(U_t, O_t), r_t^1(U_t, O_t), \ldots, r_t^L(U_t, O_t)]'$, where $r_t^l(U_t, O_t)$ is the return rate on the lth ($l = 0, 1, \ldots, L$) risky asset given the unknown market state U_t and the known market state O_t at time t. In the rest of this paper, we assume that the risky assets are non-degenerated. This means that, for $t = 0, 1, \ldots, T-1$, $\mathrm{E}[\mathbf{r}_t(U_t, O_t)] \neq \overrightarrow{0}$ and $\mathrm{E}[\mathbf{r}_t(U_t, O_t)\mathbf{r}_t'(U_t, O_t)] = \mathrm{Cov}[\mathbf{r}_t(U_t, O_t)] + \mathrm{E}[\mathbf{r}_t(U_t, O_t)]\mathrm{E}[\mathbf{r}_t'(U_t, O_t)]$ is positive definite, where $\overrightarrow{0}$ is an n dimension zero vector.

Let π_t^l denote the amount of money invested in the lth risky asset at the beginning of time period $t+1$, and let W_t denote the wealth of the DC pension fund at time t. At the beginning of period $t+1$, integrating the contribution amount $c_t(O_t)s_t$ into the pension fund account, the amount invested in the 0th risky asset over time period $t+1$ is equal to $\mathrm{W}_t + c_t(O_t)s_t - \sum_{l=1}^{L}\pi_t^l$. Hence, the wealth process $\mathrm{W} = \{\mathrm{W}_t, t = 0, 1, \ldots, T\}$ under the portfolio strategy $\pi_t = [\pi_t^1, \pi_t^2, \ldots, \pi_t^L]'$ evolves as

$$\begin{aligned}\mathrm{W}_{t+1} &= r_t^0(U_t, O_t)\left(\mathrm{W}_t + c_t(O_t)s_t - \sum_{l=1}^{L}\pi_t^l\right) \\ &\quad + \sum_{l=1}^{L} r_t^l(U_t, O_t)\pi_t^l \\ &= r_t^0(U_t, O_t)\Big(\mathrm{W}_t + c_t(O_t)s_t\Big) + \mathrm{P}_t'(U_t, O_t)\pi_t. \end{aligned} \tag{2}$$

Here, $\mathrm{P}_t(U_t, O_t) = [r_t^1(U_t, O_t) - r_t^0(U_t, O_t), r_t^2(U_t, O_t) - r_t^0(U_t, O_t), \ldots, r_t^L(U_t, O_t) - r_t^0(U_t, O_t)]'$. In addition, $\mathbf{r}_t(U_t, O_t)$ and $\nu_t(U_t, O_t)$ are supposed to be statistically independent for $t = 0, 1, \ldots, T-1$. The investment strategy $\pi := \{\pi_t, t = 0, 1, \ldots, T-1\}$ is called admissible if π_t is measurable with respect to $\Im_t^O$. Let Ξ denote the set of all admissible strategies.

We consider an investment decision-maker of a DC pension fund who has a stochastic salary, contributes a certain amount to the pension fund account, and allocates her/his wealth among these risky assets based on all of the known information about the financial market. The decision-maker predetermines a certain level of wealth to maintain an adequate income after retirement. Based on the known information, the decision-maker seeks an admissible investment strategy π such that the portfolio risk (measured by the variance of the terminal wealth) is minimized given the expected return level (measured by the expectation of the terminal wealth). The optimal investment management problem of this DC pension fund can be formulated as follows

$$\begin{cases} \min\limits_{\pi} \ \mathrm{Var}_0[\mathrm{W}_T] = \mathrm{E}_0[\mathrm{W}_T^2] - \zeta^2 \\ \text{s.t.} \ \ \mathrm{E}_0[\mathrm{W}_T] = \zeta, \\ \qquad \mathrm{W}_{t+1} = r_t^0(U_t, O_t)\Big(\mathrm{W}_t + c_t(O_t)s_t\Big) + \mathrm{P}_t'(U_t, O_t)\pi_t, \\ \qquad s_{t+1} = \nu_t(U_t, O_t)s_t, \quad t = 0, 1, \ldots, T-1 \end{cases} \tag{3}$$

where $\mathrm{E}_0[\cdot] = \mathrm{E}[\cdot|I_0, \mathrm{W}_0]$ and $\mathrm{Var}_0[\cdot] = \mathrm{Var}[\cdot|I_0, \mathrm{W}_0]$. We further assume that short selling is allowed. The solution of problem (3) is called an efficient investment strategy, and the point $(\mathrm{Var}_0[\mathrm{W}_T], \zeta)$ corresponding to an efficient investment strategy on the variance–mean space is called an efficient point. The collection of all the efficient points forms the efficient frontier in the variance–mean space.

Remark 2.1. Considering multi-period optimal investment problems for DC pension funds, Yao et al. (2014), Wu and Zeng (2015), and Yao et al. (2016b) studied the pre-commitment investment strategy and the time-consistent investment strategy under the mean–variance criterion. However, they assumed that the return rates of risky assets over each period are independent and identically distributed, and that all of the information about the financial markets can be observed at any time. Compared with the above existing literature, the most significant feature of the model (3) in this paper is the introduction of imperfectly observed market information into the multi-period mean–variance DC pension fund management model problem. As the optimal investment model with imperfect information, (3), it generalizes the existing literature on the DC pension fund management problem in following ways: (i) It introduces a generalized financial market with both observable and unobservable market states to modulate the return rate of financial assets, bringing the optimization problem closer to investment practice in real financial markets; (ii) It assumes that the salary growth rate depends on the unobservable and observable market states, rather than setting the salary as a constant or exogenous random variable unrelated to the state of the financial market. As real salary growth rates usually fluctuate with the macroeconomic environment, it is more reasonable to assume that the salary growth rate varies with the state of the financial market. (iii) Only risky assets are considered in our optimization model (3). That is because, in the long run, almost all of the financial assets are uncertainty, and the return rates of the financial assets depend on the macroeconomic environment states. On the other hand, the optimization model (3) can reduce into the one with one risk-free asset (such as bank account) when the 0th asset is set to be a constant, i.e. $r_t^0(U_0, O_0) = r_t^0$.

As U_t is unobservable, this implies that problem (3) is an optimization problem with imperfect information that cannot be solved directly by the dynamic optimization method. According to the well-known separation principle for dealing with the optimization problem with partially observable information (Wonham, 1968; Detemple, 1991), the initial optimization problem under imperfect information (3) can be solved in two steps. The first step is an inference problem in which an equivalent statistic is calculated to replace the unobservable market state at that period, transforming the optimization problem with imperfect information into one with complete information. The second step, based on the aforementioned estimated model parameter, adopts the stochastic control theory to solve the optimal control problem. In Section 3, an equivalent statistic is calculated and the optimization problem with imperfect information is transformed into one with complete information, after which Section 4 deals with the second step of optimal control.

3. The portfolio optimization problem with complete information

At the beginning of each period, decision-makers usually update their observed information set with the latest market information and estimate the model parameters based on the observed information up to the decision-making moment. Use

$$\varphi_{t-1}(i) = \Pr\{U_{t-1} = i|\overline{O}_{t-1}\}$$

to denote the conditional probability of the unobservable market state at time $t-1$, $U_{t-1} = i$, given the observation history $\overline{O}_{t-1}$ at time $t-1$. Let $\Phi(t-1) = [\varphi_{t-1}(1), \varphi_{t-1}(2), \ldots, \varphi_{t-1}(n)]'$ be the conditional probability distribution of U_{t-1} given $\overline{O}_{t-1}$ at time $t-1$. $\Phi(t-1) \in IV_n := \left\{x = [x_1, x_2, \ldots, x_n]' \in R^n : \sum_{i=1}^{n} x_i = 1, x_i \geq 0, i = 1, 2, \ldots, n\right\}$ is usually called the information vector. At the next time point t, the unobservable market state changes to U_t from U_{t-1}, and the observation process updates from $\overline{O}_{t-1}$ to $\overline{O}_t = (\overline{O}_{t-1}, O_t)$. In particular, the conditional probability of $U_t = i$, given the updated observation at time t, $\overline{O}_t = (\overline{O}_{t-1}, O_t = j)$, is specified as follows Monahan (1982),

$$\begin{aligned}\varphi_t^j(i) &= \Pr\left\{U_t = i|\overline{O}_t\right\} = \Pr\left\{U_t = i|\overline{O}_{t-1}, O_t = j\right\} \\ &= \frac{\Pr\left\{O_t = j|U_t = i, \overline{O}_{t-1}\right\}\Pr\left\{U_t = i|\overline{O}_{t-1}\right\}}{\sum_{i=1}^n \Pr\left\{O_t = j|U_t = i, \overline{O}_{t-1}\right\}\Pr\left\{U_t = i|\overline{O}_{t-1}\right\}} \\ &= \frac{\Pr\left\{O_t = j|U_t = i\right\}\sum_{l=1}^n \Pr\left\{U_t = i|U_{t-1} = l\right\}\Pr\left\{U_{t-1} = l|\overline{O}_{t-1}\right\}}{\sum_{i=1}^n \Pr\left\{O_t = j|U_t = i\right\}\sum_{l=1}^n \Pr\left\{U_t = i|U_{t-1} = l\right\}\Pr\left\{U_{t-1} = l|\overline{O}_{t-1}\right\}} \\ &= \frac{\delta_t(i,j)\sum_{l=1}^n \varphi_{t-1}(l)q_{t-1}(l,i)}{\sum_{i=1}^n \delta_t(i,j)\sum_{l=1}^n \varphi_{t-1}(l)q_{t-1}(l,i)} = \beta_t(\Phi(t-1), i, j). \end{aligned} \tag{4}$$

According to Eq. (4), the conditional probability distribution of the unobservable market state, $\varphi_t(i)$, can be inferred on the basis of the information vector of the last time period, $\varphi_{t-1}(l)$, and the observed market state (updated observation) at time t, $O_t = j$. In other words, $\Phi^j(t)$ summarizes all the necessary information for making decisions at time t and can completely substitute for the unobservable state variables U_t (Monahan, 1982; Detemple, 1991). Moreover, $\Phi^j(t) = \{\varphi_t^j(1), \varphi_t^j(2), \dots, \varphi_t^j(n)\}$ is the equivalent statistic for the unobservable market state at time t. Note that the conditional probability distribution of the unobservable market state on the given observed market state $O_0 = j$ at the initial time, $\Phi^j(0) = \{\varphi_0^j(1), \varphi_0^j(2), \dots, \varphi_0^j(n)\}$, can be calculated by

$$\varphi_0^j(i) = \frac{\Pr\{U_0 = i\}\delta_0(i,j)}{\sum_{i=1}^n \Pr\{U_0 = i\}\delta_0(i,j)}. \tag{5}$$

In practice, $\Pr\{U_0 = i\}$ is often estimated by the decision-maker based on the preliminary analysis of the unobservable market state at the initial time.

Monahan (1982) states that the information vector process $\{\Phi^j(t), 0 \le t \le T, 1 \le j \le m\}$ is a Markov chain where $\Phi^j(t)$ is the state of this Markov chain at time t. As $\Phi^j(t)$ is an equivalent statistic of the unobserved state variable, and the number of $\Phi^j(t)$ is the same as the state number of U_t, we can replace U_t with $\Phi^j(t)$, which converts the partially observable Markov decision process into an equivalent (completely observable) Markov decision process. Now the optimization problem under imperfect information (3) can be transformed into one with complete information as follows:

$$\begin{cases} \min\limits_{\pi} \ \mathrm{Var}[W_T|\Phi^{O_0}(0), O_0, W_0] = \mathrm{E}[W_T^2|\Phi^{O_0}(0), O_0, W_0] - \zeta^2 \\ \text{s.t.} \ \ \mathrm{E}[W_T|\Phi^{O_0}(0), O_0, W_0] = \zeta, \\ \qquad W_{t+1} = r_t^0(\Phi^j(t), j)\Big(W_t + c_t(j)s_t\Big) + P_t'(\Phi^j(t), j)\pi_t, \\ \qquad s_{t+1} = \nu_t(\Phi^j(t), j)s_t, \quad O_t = j \ \text{ for} \\ \qquad \quad t = 0, 1, \dots, T-1, j = 1, 2, \dots, m. \end{cases} \tag{6}$$

This completes the first step of estimating the model parameters and transforming an optimization problem with imperfect information into one with complete information.

4. The optimal investment strategy and efficient frontier

In problem (6), the equation constraint $\mathrm{E}[W_T|\Phi^{O_0}(0), O_0, W_0] = \zeta$ can be removed using the Lagrange method. For the given fixed Lagrange multiplier $2a$, we can solve the equivalent optimization problem (6) as follows.

$$\begin{cases} \min\limits_{\pi} \ \mathrm{E}[W_T^2|\Phi^{O_0}(0), O_0, W_0] - \zeta^2 \\ \qquad - 2a\Big(\mathrm{E}[W_T|\Phi^{O_0}(0), O_0, W_0] - \zeta\Big) \\ \text{s.t.} \ \ W_{t+1} = r_t^0(\Phi^j(t), j)\Big(W_t + c_t(j)s_t\Big) + P_t'(\Phi^j(t), j)\pi_t, \\ \qquad s_{t+1} = \nu_t(\Phi^j(t), j)s_t, \quad t = 0, 1, \dots, T-1, \\ \qquad \quad j = 1, 2, \dots, m. \end{cases} \tag{7}$$

As

$$\begin{aligned} &\mathrm{E}[W_T^2|\Phi^{O_0}(0), O_0, W_0] - \zeta^2 - 2a\Big(\mathrm{E}[W_T|\Phi^{O_0}(0), O_0, W_0] - \zeta\Big) \\ &\quad = \mathrm{E}\Big[W_T^2 - 2aW_T|\Phi^{O_0}(0), O_0, W_0\Big] - \zeta^2 + 2a\zeta, \end{aligned} \tag{8}$$

and $-\zeta^2 + 2a\zeta$ is deterministic, optimization problem (7) can be written as

$$\begin{cases} \min\limits_{\pi} \ \mathrm{E}\Big[W_T^2 - 2aW_T|\Phi^{O_0}(0), O_0, W_0\Big] \\ \text{s.t.} \ \ W_{t+1} = r_t^0(\Phi^j(t), j)\Big(W_t + c_t(j)s_t\Big) + P_t'(\Phi^j(t), j)\pi_t, \\ \qquad s_{t+1} = \nu_t(\Phi^j(t), j)s_t, \quad t = 0, 1, \dots, T-1, \\ \qquad \quad j = 1, 2, \dots, m. \end{cases} \tag{9}$$

Because optimization problem (9) possesses the same optimal investment strategy as optimization problem (7), we first solve optimization problem (9).

For $t = 0, 1, \dots, T-1$ and $j = 1, 2, \dots, m$, let

$$V_t^*(\Phi^j(t), j, W_t) = \min_{\pi} \ \ \mathrm{E}\Big[W_T^2 - 2aW_T|\Phi^j(t), j, W_t\Big] \tag{10}$$

denote the optimal expected utility under the optimal decision from time t to time $T-1$ given the observed market state at time t, $O_t = j$, the estimation probability distribution of the unobservable market state at time t, $\Phi^j(t)$, and the amount of wealth at time t, W_t. In the rest of this paper, in order to analysis the impact of the observed market state, we particularly denote the investment strategy over period $t+1$ by $\pi_t(j)$ given the observed market state $O_t = j$.

According to Bellman's principle of optimality, we have the following Bellman equation for optimization problem (9).

$$\begin{aligned} &V_t^*(\Phi^j(t), j, W_t) \\ &= \min_{\pi_t(j)} \mathrm{E}\Big[V_{t+1}^*(\Phi(t+1), O_{t+1}, W_{t+1})|\Phi^j(t), j, W_t\Big] \\ &= \min_{\pi_t(j)} \sum_{i=1}^n \varphi_t^j(i)\mathrm{E}\Big[V_{t+1}^*(\Phi(t+1), O_{t+1}, W_{t+1})\Big] \\ &= \min_{\pi_t(j)} \sum_{i=1}^n \varphi_t^j(i)\sum_{k=1}^m \theta_t(i,k)\mathrm{E}\Big[V_{t+1}^*(\Phi^k(t+1), k, r_t^0(\Phi^j(t), j) \\ &\qquad \times (W_t + c_t(j)s_t) + P_t'(\Phi^j(t), j)\pi_t)\Big] \end{aligned} \tag{11}$$

where $\theta_t(i,k) = \sum_{l=1}^n q_t(i,l)\delta_{t+1}(l,k)$, with boundary condition $V_T^*(\Phi^j(T), j, W_T) = W_T^2 - 2aW_T$.

By solving (11) recursively, we can obtain the optimal value function and the optimal solution for problem (9), both of which are summarized in the following theorem.

Theorem 4.1. *The optimal value function of optimization problem (9), namely the solution to the Bellman equation (11), is specified by*

$$\begin{aligned} V_t^*(\Phi^j(t), j, W_t) &= -a^2B_t(j) - 2a\Big[D_t^1(j)W_t + D_t^2(j)s_t\Big] \\ &\quad + A_t^1(j)(W_t)^2 + 2A_t^2(j)W_ts_t + A_t^3(j)(s_t)^2, \end{aligned} \tag{12}$$

and the optimal investment strategy is given by

$$\pi_t^*(j) = a\sigma_t^{-1}(j)h_t(j) - \sigma_t^{-1}(j)\Big[\eta_t(j)W_t + u_t(j)s_t\Big] \tag{13}$$

where $h_t(j), \eta_t(j), u_t(j), \sigma_t(j), B_t(j), D_t^{(1)}(j), D_t^{(2)}(j), A_t^{(1)}(j), A_t^{(2)}(j), A_t^{(3)}(j)$ are given in (A.6) to (A.14).

Proof. See Appendix A.

Lemma 4.1. *For $t = 0, 1, \dots, T-1$ and $j = 1, 2, \dots, m$, we have $A_t^{(1)}(j) > 0$.*

Proof. See Appendix B.

Lemma 4.2. *Suppose that for $i = 1, 2, \ldots, m$, τ_i is a nonnegative constant which satisfies $0 \le \tau_i \le 1$ and $\sum_{i=1}^{n}\tau_i = 1$. We assume that $\xi_i = [\xi_i^1, \xi_i^2, \ldots, \xi_i^n]'$ is a non-degenerate random vector, i.e. $\mathrm{E}[\xi_i] = \left[\mathrm{E}[\xi_i^1], \mathrm{E}[\xi_i^2], \ldots, \mathrm{E}[\xi_i^n]\right]' \neq \overrightarrow{0}$, and $\Sigma_i = \left[\mathrm{Cov}(\xi_{i_1}, \xi_{i_2})\right]_{n\times n}$ is positive definite where $1 \le i_1, i_2 \le n$. Then we have*

$$0 < \left[\sum_{i=1}^{n}\tau_i \mathrm{E}[\xi_i]\right]'\left[\sum_{i=1}^{n}\tau_i \mathrm{E}[\xi_i\xi_i']\right]^{-1}\left[\sum_{i=1}^{n}\tau_i \mathrm{E}[\xi_i]\right] < 1. \tag{14}$$

Proof. See Appendix B.

Lemma 4.3. *For $t = 0, 1, \ldots, T-1$ and $j = 1, 2, \ldots, m$, we have $0 < B_t(j) < 1$.*

Proof. See Appendix C.

From the relationship between the solutions to optimization problem (6) and optimization problem (9), we notice that the optimal value for problem (6) can be given by

$$\begin{aligned} f_0(\Phi^j(0), j, \mathrm{W}_0) &= \mathrm{V}_t^*(\Phi^j(0), j, \mathrm{W}_0) - \zeta^2 + 2a\zeta \\ &= -a^2\mathrm{B}_0(j) - 2a\left[D_0^{(1)}(j)\mathrm{W}_0 + D_0^{(2)}(j)s_0 - \zeta\right] \\ &\quad + A_0^{(1)}(j)(\mathrm{W}_0)^2 + 2A_0^{(2)}(j)\mathrm{W}_0 s_0 \\ &\quad + A_0^{(3)}(j)(s_0)^2 - \zeta^2. \end{aligned} \tag{15}$$

According to the Lagrange dual theory (see Luenberger, 1968), the optimal value of the objective for problem (6) can be reached by maximizing $f_0(\Phi^j(0), j, \mathrm{W}_0)$ over a, i.e.

$$\mathrm{Var}_{0,j}[\mathrm{W}_T] = \max_a \quad f_0(\Phi^j(0), j, \mathrm{W}_0). \tag{16}$$

Notice that $B_0(j) > 0$, which implies that an optimal solution of optimization problem (16) exists. By using the first order condition, we obtain the optimal solution of problem (16) given by

$$a^* = \frac{\zeta - [D_0^{(1)}(j)\mathrm{W}_0 + D_0^{(2)}(j)s_0]}{B_0(j)}. \tag{17}$$

Substituting (17) back into (13), we obtain the optimal investment strategy of the optimization model under the mean–variance criterion (6), namely the efficient investment strategy, as follows

$$\begin{aligned} \pi_t^*(j) &= -\sigma_t^{-1}(j)\eta_t(j)\mathrm{W}_t - \sigma_t^{-1}(j)u_t(j)s_t \\ &\quad + \frac{\zeta - [D_0^{(1)}(j)\mathrm{W}_0 + D_0^{(2)}(j)s_0]}{B_0(j)}\sigma_t^{-1}(j)h_t(j). \end{aligned} \tag{18}$$

By also substituting (17) into (15), we obtain the efficient frontier of optimization problem (6) as

$$\begin{aligned} \mathrm{Var}_{0,j}[\mathrm{W}_T] &= \frac{1 - B_0(j)}{B_0(j)}\left[\mathrm{E}_{0,j}[\mathrm{W}_T] - \frac{D_0^{(1)}(j)\mathrm{W}_0 + D_0^{(2)}(j)s_0}{1 - B_0(j)}\right]^2 \\ &\quad + \left[A_0^{(1)}(j) - \frac{(D_0^{(1)}(j))^2}{1 - B_0(j)}\right](\mathrm{W}_0)^2 \\ &\quad + 2\left[A_0^{(2)}(j) - \frac{D_0^{(1)}(j)D_0^{(2)}(j)}{1 - B_0(j)}\right]\mathrm{W}_0 s_0 \\ &\quad + \left[A_0^{(3)}(j) - \frac{(D_0^{(2)}(j))^2}{1 - B_0(j)}\right](s_0)^2. \end{aligned} \tag{19}$$

Theorem 4.2. *For a pregiven expected level of terminal wealth $\mathrm{E}_{0,j}[\mathrm{W}_T] = \zeta(\zeta \ge \frac{D_0^{(1)}(j)\mathrm{W}_0 + D_0^{(2)}(j)s_0}{1 - B_0(j)})$, the optimal investment strategy and efficient frontier of the multi-period investment problem for a mean–variance DC pension fund (6) are given by (18) and (19), respectively.*

Let $\epsilon_t(j) = \sigma_t^{-1}(j)\eta_t(j)$, $\tau_t(j) = \sigma_t^{-1}(j)u_t(j)$ and $\chi_t(j) = \sigma_t^{-1}(j)h_t(j)$. Then the optimal investment strategy (18) can be rewritten in the following concise form:

$$\pi_t^*(j) = -\epsilon_t(j)\mathrm{W}_t - \tau_t(j)s_t + \frac{\zeta - [D_0^{(1)}(j)\mathrm{W}_0 + D_0^{(2)}(j)s_0]}{B_0(j)}\chi_t(j). \tag{20}$$

Clearly, the efficient investment strategy (20) is an affine function of the current wealth W_t and the current salary s_t. Moreover, (20) discloses the "three-fund" property of the optimal strategy. Here, $\hat{\epsilon}_t(j) = \frac{\epsilon_t(j)}{\mathbf{1}'\cdot\epsilon_t(j)}$, $\hat{\tau}_t(j) = \frac{\tau_t(j)}{\mathbf{1}'\cdot\tau_t(j)}$ and $\hat{\chi}_t(j) = \frac{\chi_t(j)}{\mathbf{1}'\cdot\chi_t(j)}$ can be viewed as three funds, where $\mathbf{1}$ is the n dimension all-one vector. The optimal investment strategy is a linear combination of three mutual funds: $\hat{\epsilon}_t(j)$, $\hat{\tau}_t(j)$, and $\hat{\chi}_t(j)$. Hence, in a financial market consisting of $n+1$ risky assets, investors need only keep a balance among four financial assets: the 0th risky asset and three "artificial" mutual funds, $\hat{\epsilon}_t(j)$, $\hat{\tau}_t(j)$, and $\hat{\chi}_t(j)$, according to the current wealth level W_t, current salary level s_t, current observed market information $O_t = j$, and initial variable (W_0, s_0). Even for a pure investment problem, which means that $s_t = 0$, the optimal investment strategy is a summation of two portfolios, one proportional to $\hat{\epsilon}_t(j)$, and the other proportional to $\hat{\chi}_t(j)$. Notice that there is also a kind of "three-fund" and "two-fund" property in the discrete-time portfolio selection problem under the mean–variance framework with full information; see for example Yao et al. (2016b). In the continuous-time portfolio selection problem, Munk and Sørensen (2004) also get an analogous three-fund theorem in a setting with stochastic income.

Notice that the parameter $B_0(j)$ in (19) is only determined by the observable and unobservable market information and the return rate of the financial assets. As proved by Zhang et al. (2016), $B_0(j)$ reflects the investment value of the risky assets from time 0 to time T. The introduction of unobservable market information reduces the investment value of the risky assets compared with the case of completely observable market information, on average. As $B_0(j)$ decreases, it reduces the investment value of the risky assets. $\mathrm{Var}_{0,j}[\mathrm{W}_T]$ increases correspondingly, meaning that the investment risk for DC pension funds rises. This result is consistent with our intuition. When less market information is observed by the investor, the investment risk correlated with the expected excess return on risky assets is increased, and risky assets become less attractive to the investor.

Letting $\zeta_{\sigma_{\min}} = \frac{D_0^{(1)}(j)\mathrm{W}_0 + D_0^{(2)}(j)s_0}{1 - B_0(j)}$, we receive the global minimum variance

$$\begin{aligned} \mathrm{Var}_{0,j}^{\min}[\mathrm{W}_T] &= \left[A_0^{(1)}(j) - \frac{(D_0^{(1)}(j))^2}{1 - B_0(j)}\right](\mathrm{W}_0)^2 \\ &\quad + 2\left[A_0^{(2)}(j) - \frac{D_0^{(1)}(j)D_0^{(2)}(j)}{1 - B_0(j)}\right]\mathrm{W}_0 s_0 \\ &\quad + \left[A_0^{(3)}(j) - \frac{(D_0^{(2)}(j))^2}{1 - B_0(j)}\right](s_0)^2. \end{aligned} \tag{21}$$

5. Some special cases

The hidden Markov model proposed in the last section is a generalized situation for the mean–variance optimal investment management of a DC pension fund. In this section, we compare and discuss some special cases of our hidden Markov model.

5.1. The case with one risk-free asset

First, we consider the case with one risk-free asset at each time period, which implies that $r_t^0(U_t, O_t) = r_t^0$ is constant over time period $t+1$. Meanwhile, the salary growth rate and contribution rate are supposed to be independent of the financial market states, i.e. $\nu_t(i,j)=\nu_t$, $c_t(i)=c_t$ for $t=1,2,\ldots,T-1$, $i=1,2,\ldots,n$ and $j=1,2,\ldots,m$. Therefore, in this case, the model parameters (A.6)–(A.14) reduce to

$$h_t(j)=\eta_t(j)=\sum_{i=1}^{n}\Big[\varphi_t^j(i)\mathrm{E}[\mathrm{P}_t(i,j)]\times\sum_{k=1}^{m}\theta_t(i,k)\left(1-\mathrm{B}_{t+1}(k)\right)\Big], \tag{22}$$

$$u_t(j)=\left[\sum_{q=t}^{T-1}\left(\prod_{s=t}^{q-1}\mathrm{E}[\nu_s]\right)(r_p^0c_p)\right]h_t(j), \tag{23}$$

$$\sigma_t(j)=\sum_{i=1}^{n}\Big[\varphi_t^j(i)\mathrm{E}[\mathrm{P}_t(i,j)\mathrm{P}_t'(i,j)]\sum_{k=1}^{m}\theta_t(i,k)(1-\mathrm{B}_{t+1}(k))\Big], \tag{24}$$

$$B_t(j)=h_t'(j)\sigma_t^{-1}(j)h_t(j)+\sum_{i=1}^{n}\Big[\varphi_t^j(i)\sum_{k=1}^{m}\theta_t(i,k)\mathrm{B}_{t+1}(k)\Big], \tag{25}$$

$$D_t^{(1)}(j)=A_t^{(1)}(j)=1-B_t(j), \tag{26}$$

$$D_t^2(j)=A_t^{(2)}(j)=\left[\sum_{q=t}^{T-1}\left(\prod_{s=t}^{q-1}\mathrm{E}[\nu_s]\right)(r_p^0c_p)\right](1-B_t(j)), \tag{27}$$

$$A_t^{(3)}(j)=\left[\sum_{q=t}^{T-1}\left(\prod_{s=t}^{q-1}\mathrm{E}[\nu_s]\right)(r_p^0c_p)\right]^2(1-B_t(j))+\kappa_t(j), \tag{28}$$

$$\begin{aligned}\kappa_t(j)=&\left[\sum_{q=t+1}^{T-1}\left(\prod_{s=t+1}^{q-1}\mathrm{E}[\nu_s]\right)(r_p^0c_p)\right]^2\\&\times\left[\sum_{i=1}^{n}\varphi_t^j(i)\sum_{k=1}^{m}\theta_t(i,k)\left(1-\mathrm{B}_{t+1}(k)\right)\right]\mathrm{Var}[\nu_t]\\&+\sum_{i=1}^{n}\Big[\varphi_t^j(i)\sum_{k=1}^{m}\theta_t(i,k)\kappa_{t+1}(k)\Big],\quad \kappa_{T-1}(j)=0.\end{aligned} \tag{29}$$

The optimal investment strategy (18) then reduces to

$$\pi_t^{\mathrm{MV}}(j)=\left\{\begin{array}{c}\left(\prod_{s=t}^{T-1}r_s^0\right)\mathrm{W}_t-\left[\sum_{q=t}^{T-1}\left(\prod_{s=t}^{q-1}\mathrm{E}[\nu_s]\right)(r_p^0c_p)\right]s_t+\\ \dfrac{\zeta-(1-B_0(j))\left\{\left(\prod_{t=0}^{T-1}r_t^0\right)\mathrm{W}_0-\left[\sum_{q=0}^{T-1}\left(\prod_{s=0}^{q-1}\mathrm{E}[\nu_s]\right)(r_p^0c_p)\right]s_0\right\}}{B_0(j)}\end{array}\right\}\times\sigma_t^{-1}(j)h_t(j), \tag{30}$$

and the efficient frontier is given by

$$\mathrm{Var}_{0,j}[\mathrm{W}_T]=\frac{1-B_0(j)}{B_0(j)}\left\{\mathrm{E}_{0,j}[\mathrm{W}_T]-\left[\left(\prod_{t=0}^{T-1}r_t^0\right)\mathrm{W}_0-\left[\sum_{q=0}^{T-1}\left(\prod_{s=0}^{q-1}\mathrm{E}[\nu_s]\right)(r_p^0c_p)\right]s_0\right]\right\}^2+\kappa_0(j)s_0^2. \tag{31}$$

Let $\psi_t=\sigma_t^{-1}(j)h_t(j)$ and $\hat{\psi}_t=\frac{\psi_t}{\mathbf{1}'\psi_t}$, where $\mathbf{1}$ is the n dimensional all-one vector. Then, the expression of the efficient investment strategy $\pi_t^{\mathrm{MV}}(j)$ for DC pension funds is proportional to ψ_t, which is similar to the optimal investment strategy under the Markov regime-switching model given in Çakmak and Özekici (2006). The expression (30) also implies that the well-known one-fund theorem in the classical dynamic portfolio choice problem (Li and Ng, 2000) with complete information also holds true. Compared with the optimal investment strategy (20), when there is a risk-free asset (i.e. $r_t^0(U_t,O_t)=r_t^0$), all three funds, $\hat{\epsilon}_t(j)$, $\hat{\tau}_t(j)$, and $\hat{\chi}_t(j)$, reduce to $\hat{\psi}_t(j)$. Correspondingly, the optimal investment strategy (20) is proportional to $\hat{\psi}_t(j)$, which goes back to the one-fund theorem. Therefore, with one risk-free asset and n risky assets in the financial market, the decision-maker for a DC pension fund only needs to balance the amount of the pension fund account among the risk-free asset and one "artificial" mutual fund, $\hat{\psi}_t(j)$, according to the current observed market information, current and initial wealth levels, current and initial salary levels, and expected return level. In particular, for a pure portfolio selection problem, i.e. $s_0=s_1=\cdots=s_T=0$, the optimal strategy (20) reduces to the one in Li and Ng (2000). In this sense, we extend the multi-period mean–variance portfolio problem of Li and Ng (2000) with IID asset returns into one with a stochastic salary and partially observed information.

However, by comparing the efficient frontiers (19) and (31), we find that introducing the risk-free asset simplifies the structure of the global minimum variance of the portfolio optimization problem for DC pension funds. For the pure investment problem in particular, i.e. $s_0=0$, the efficient frontier is reduced to the following expression

$$\mathrm{Var}_{0,j}[\mathrm{W}_T]=\frac{1-B_0(j)}{B_0(j)}\left[\mathrm{E}_{0,j}[\mathrm{W}_T]-\left(\prod_{t=0}^{T-1}r_t^0\right)\mathrm{W}_0\right]^2 \tag{32}$$

and

$$\sigma_{0,j}[\mathrm{W}_T]=\sqrt{\mathrm{Var}_{0,j}[\mathrm{W}_T]}=\frac{1-B_0(j)}{B_0(j)}\left[\mathrm{E}_{0,j}[\mathrm{W}_T]-\left(\prod_{t=0}^{T-1}r_t^0\right)\mathrm{W}_0\right], \tag{33}$$

which is a radial line emitted from point $\left(0,\left(\prod_{t=0}^{T-1}r_t^0\right)\mathrm{W}_0\right)$ with slope $\frac{1-B_0(j)}{B_0(j)}$ in the standard derivation-mean plane. In fact, drawing an analogy with a pure self-financing investment problem, the optimal investment management problem for DC pension funds can be viewed as an investment problem with stochastic capital flow at the beginning of each period. Therefore, a liquidity risk emerges in the investment management of DC pension funds compared with a pure self-financing investment problem. As $\kappa_0(j)>0$ for $j=1,2,\ldots,m$, we have $\mathrm{Var}_{0,j}^{\min}[\mathrm{W}_T]=\kappa_0(j)s_0^2>0$, which implies that we cannot find a portfolio strategy to hedge the investment risk due to the intake of stochastic capital flow. From this point, the liquidity risk cannot be fully hedged by the market assets, and ignoring the liquidity risk may lead to investment loss.

5.2. The case of completely observable information

When the state of the financial market at each time period can be completely observed, we have $U_t=O_t$ for $t=0,1,\ldots,T$ and the hidden Markov chain U becomes a Markov chain. Therefore, for a given $U_t=i$,

$$\delta_t(i,j)=Pr\{O_t=j|U_t=i\}=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\neq j,\end{cases} \tag{34}$$

leading to $\Delta_t=I_n$, which means Δ_t reduces to an $n\times n$ identical matrix. Furthermore,

$$\Phi^j(t,i)=Pr\{U_t=i|\overline{O}_{t-1},O_t=j\}=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\neq j.\end{cases} \tag{35}$$

And $\theta_t(j,k)=\sum_{l=1}^{n}q_t(j,l)\delta_t(l,k)=q_t(j,k)$. At time t under conditions of observed market state $O_t=j$, the excess return vector of risky assets $\mathrm{P}_t(i,j)$ reduces to $\mathrm{P}_t(j)$. Therefore, (22)–(29)

further reduces to

$$h_t(j) = \eta_t(j) = \mathrm{E}[\mathrm{P}_t(i,j)]\left(1 - \sum_{k=1}^{m} q_t(i,k)\mathrm{B}_{t+1}(k)\right), \tag{36}$$

$$u_t(j) = \left[\sum_{q=t}^{T-1}\left(\prod_{s=t}^{q-1}\mathrm{E}[\nu_s]\right)(r_p^0 c_p)\right] h_t(j), \tag{37}$$

$$\sigma_t(j) = \mathrm{E}[\mathrm{P}_t(i,j)\mathrm{P}_t'(i,j)]\left(1 - \sum_{k=1}^{m} q_t(i,k)\mathrm{B}_{t+1}(k)\right), \tag{38}$$

$$B_t(j) = h_t'(j)\sigma_t^{-1}(j)h_t(j) + \sum_{k=1}^{m} q_t(i,k)\mathrm{B}_{t+1}(k), \tag{39}$$

$$D_t^{(1)}(j) = A_t^{(1)}(j) = 1 - B_t(j), \tag{40}$$

$$D_t^{(2)}(j) = A_t^{(2)}(j) = \left[\sum_{q=t}^{T-1}\left(\prod_{s=t}^{q-1}\mathrm{E}[\nu_s]\right)(r_p^0 c_p)\right](1 - B_t(j)), \tag{41}$$

$$A_t^{(3)}(j) = \left[\sum_{q=t}^{T-1}\left(\prod_{s=t}^{q-1}\mathrm{E}[\nu_s]\right)(r_p^0 c_p)\right]^2 (1 - B_t(j)) + \kappa_t(j), \tag{42}$$

$$\begin{aligned}\kappa_t(j) = &\left[\sum_{q=t+1}^{T-1}\left(\prod_{s=t+1}^{q-1}\mathrm{E}[\nu_s]\right)(r_p^0 c_p)\right]^2 \\ &\times\left[\left(1 - \sum_{k=1}^{m} q_t(i,k)\mathrm{B}_{t+1}(k)\right)\right]\mathrm{Var}[\nu_t] \\ &+ \sum_{k=1}^{m} q_t(i,k)\kappa_{t+1}(k), \quad \kappa_{T-1}(j) = 0\end{aligned} \tag{43}$$

where $B_t(j)$ satisfies

$$\begin{aligned}1 - B_t(j) = &\left[\sum_{k=1}^{m} q_t(j,k)\left(1 - B_{t+1}(k)\right)\right]\mathrm{E}[\mathrm{P}_t'(j)] \\ &\times\left[\mathrm{E}[\mathrm{P}_t(j)\mathrm{P}_t'(j)]\right]^{-1}\mathrm{E}[\mathrm{P}_t(j)].\end{aligned}$$

The expressions of the optimal investment strategy and the efficient frontier are also given as (30) and (31), respectively, with the coefficients given by (36)–(43). Furthermore, if there is no pension salary, but contributions exist, i.e. $c_t = 1$, $\nu_t = 0$, our model result reduces to the case of Çakmak and Özekici (2006) where the return on the risk-free asset is constant at each period.

6. Numerical analysis

In this section, we demonstrate the impact of imperfect information on the efficient frontier and the optimal investment strategy by comparing the results between the complete information model (CIM) and the hidden Markov model (HMM). We use the Clsidx data of Shanghai's A-share index from January 4, 2012 to November 31, 2015 to determine the parameters of the hidden Markov model. By using the method of cluster analysis, we determine two unobservable market states, named the bull and bear market denoted by $i = 1, 2$. Meanwhile, we calculate the lognormal return rates of risky asset. If the lognormal return rate is greater than 0, we call it to be a positive outlook (PO). Otherwise, it is a negative outlook (NO). So we have two market observations: a positive outlook and a negative outlook. Using the Baum–Welch algorithm which can be found in Rabiner (1989), we determine the transition matrices of the hidden Markov chain U at each time period (year) as follows

$$Q_1 = \begin{bmatrix} 0.7161 & 0.2839 \\ 0.2192 & 0.7808 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.3316 & 0.6684 \\ 0.6892 & 0.3108 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 0.995 & 0.005 \\ 0.1283 & 0.8717 \end{bmatrix}, \quad Q_4 = \begin{bmatrix} 0.9616 & 0.0384 \\ 0 & 1 \end{bmatrix}.$$

During the end of 2011 to early 2012, the financial market in China keeps a declining trend. So we set the initial probability distribution of the unobservable market state to be 1 probability for bear market and 0 probability for bull market. Meanwhile, the relationship between observable and unobservable market states at each time period is evaluated as the following matrices,

$$\Delta_1 = \begin{bmatrix} 0.5680 & 0.4320 \\ 0.3273 & 0.6727 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} 0.5737 & 0.4263 \\ 0.4855 & 0.5145 \end{bmatrix},$$

$$\Delta_3 = \begin{bmatrix} 0.3246 & 0.6754 \\ 0.6925 & 0.3075 \end{bmatrix}, \quad \Delta_4 = \begin{bmatrix} 0.6298 & 0.3702 \\ 0.5637 & 0.4363 \end{bmatrix}.$$

In other words, according to Δ_1, the first unobservable market state emits a positive outlook with a probability of 0.5737 and negative outlook with a probability of 0.4320. The second unobservable market state emits a positive outlook with a probability of 0.3273 and a negative outlook with a probability of 0.6727.

Consider a DC pension fund with initial wealth $W_0 = 1$ and initial salary $s_0 = 0.2$, and the pension fund member plans to retire at time $T = 4$. At the beginning of each period, the pension fund member contributes 15 percent of her/his salary to the DC pension fund account, i.e. $c_t = 0.15$. Suppose that the 0th risky asset is a bank account with return rates $r_t^0(U_t, O_t) = 4.3\%$. We choose three stocks listed in China Stock Market with stock codes 000651, 300325, 600036 as the risky assets in our model. Using the historical daily data from January 5, 2015 to November 30, 2016, we obtain the return rates on these three risky assets under different market states (U_t, O_t) at each time period as

$$\mathrm{E}[r_t^1(i,j)] = \begin{array}{c} \begin{array}{cc} \text{PO} & \text{NO} \end{array} \\ \begin{bmatrix} 0.0335 & 0.0260 \\ -0.002 & 0.0209 \end{bmatrix} \end{array} \begin{array}{l} i = 1 \\ i = 2, \end{array}$$

$$\mathrm{E}[r_t^2(i,j)] = \begin{bmatrix} 0.0369 & 0.0206 \\ 0.0003 & -0.0038 \end{bmatrix},$$

$$\mathrm{E}[r_t^3(i,j)] = \begin{bmatrix} 0.0340 & 0.0187 \\ 0.0323 & 0.0179 \end{bmatrix},$$

for $t = 0, 1, \ldots, T-1$, $i = 1, 2$, and $j = \text{PO}, \text{NO}$. That is to say, at each time period, when the unobservable market state is $i = 1$ and the observed market state is PO, the expected return rates on the risky assets are 0.0335 for the first risky asset, 0.0369 for the second, 0.034 for the third. In addition, we set $\mathrm{E}[\nu_t(i,j)] = 1.15$ and $\mathrm{E}[\nu_t(i,j)]^2 = 1.3$ for $i = 1, 2$, $j = \text{PO}, \text{NO}$, and $t = 0, 1, \ldots, T-1$.

6.1. Analysis of the effect of imperfect information on the efficient frontier

Fig. 1 illustrates the efficient frontier of the hidden Markov model under the market observation of a positive or negative outlook. Obviously, in the hidden Markov model, the performance of the efficient frontier under the positive outlook exceeds the case of negative outlook, which is consistent with our intuition about the investment activity in the financial market. In Fig. 1, we can see that, to achieve the same expected terminal wealth, the decision-maker suffers more investment risk under a negative rather than a positive outlook market observation. Usually, when the financial market performance shows a negative rather than a positive outlook, the return rates of the risky assets become more volatile. Correspondingly, the decision-maker bears a greater investment risk under a negative outlook market. Meanwhile, the

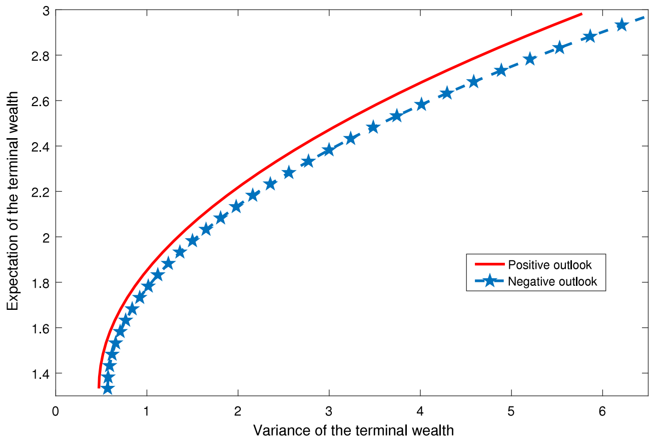

Fig. 1. Efficient frontiers for the hidden Markov model under positive and the negative outlooks.

difference of the investment risk between the positive and negative outlook becomes greater with the increasing of the expected terminal wealth.

Fig. 2 compares the efficient frontiers of the complete information model and the hidden Markov model under market performance with a positive and a negative outlook, separately. Similar to the results of the hidden Markov model, the efficient frontier of the complete information model under a positive outlook takes an advantage over a negative market observation. Under the same level of investment risk, the expected terminal wealth of the complete information model is much more greater than that of the hidden Markov model whether or not the market observation is positive or negative. This implies that, for the mean–variance DC pension fund, the more market information received, the better investment benefit achieved. Our result agrees with the investment activity in financial markets. Actually, the decision-maker of the DC pension fund stands the financial market with limited information. In order to obtain a better investment return, the decision-makers usually try their best to gather the information to correct their estimation about the returns of financial assets. The more market information is obtained, the better market state is evaluated. So the decision-maker can grab more investment opportunities and reduce the investment risk. However, under a positive outlook, the difference in the efficient frontiers between the complete information model and the hidden Markov model is smaller than that with a negative outlook. The reason may lie in that the uncertainty under a negative outlook is higher than that under a positive outlook. Hence, the volatility of the risky asset's return becomes greater and the decision-maker has to bear more investment risk.

Salary is a key factor that significantly affect the performance of the DC pension fund. So we want to analyze the impact of the salary's growth rate on the efficient frontier. If we let the other model parameters remain unchanged and vary the variance of the salary's growth rate, $\mathrm{Var}[\nu_t(i,j)]$, from 0.3 to 0.7, we obtain the efficient frontiers under the observations of both a positive and a negative outlook, as demonstrated in Fig. 3. We can find that the decision-maker endures more investment risk for the same level of expected terminal wealth when the variance of the salary growth rate increases. That is to say, the uncertainty on the growth rate leads to more investment uncertainty. Comparing the two subfigures in Fig. 3, we can conclude that the efficient frontier of the optimal DC pension fund management problem under a positive outlook exceeds that under a negative outlook.

6.2. Analysis of the optimal investment strategy

Tables 1 and 2 exhibit the investment decision-making process for both the complete information model and the hidden Markov model under the two different observation processes.

In Tables 1 and 2, for both positive and negative outlooks observation process, we find that the absolute value of the investment of the risky assets decreases as the investment activity approaches the end time point under both complete and imperfect information settings. This is consistent with real investment practice of DC pension funds in financial markets. Usually, financial consultants advise investors to reduce their investment amount in stocks when it approaches to the end of investment activity or the time of retirement. This is called "age effect". Regardless of positive or negative outlooks, the short-selling amount is greater in the case with complete information than that in the case with imperfect information, and the (positive) investment amount under imperfect information is lower than that under complete information. When the financial market shows more positive information, the decision-maker under CIM invests more money in the risky assets comparing to the case of HMM. However, if the financial market continuously emits negative information, the absolute amount invested in the risky assets under HMM is greater than that under CIM. When the decision-maker gets the complete information of the financial market, she/he can precisely forecast the future returns of the risky assets. If the market information looks more and more pessimistic, it is natural to reduce the investment amount in the risky assets to control investment risk. However, under the partially observed information, there inevitably exists estimation error and the optimal investment strategy may fluctuate up and down comparing to the case of CIM.

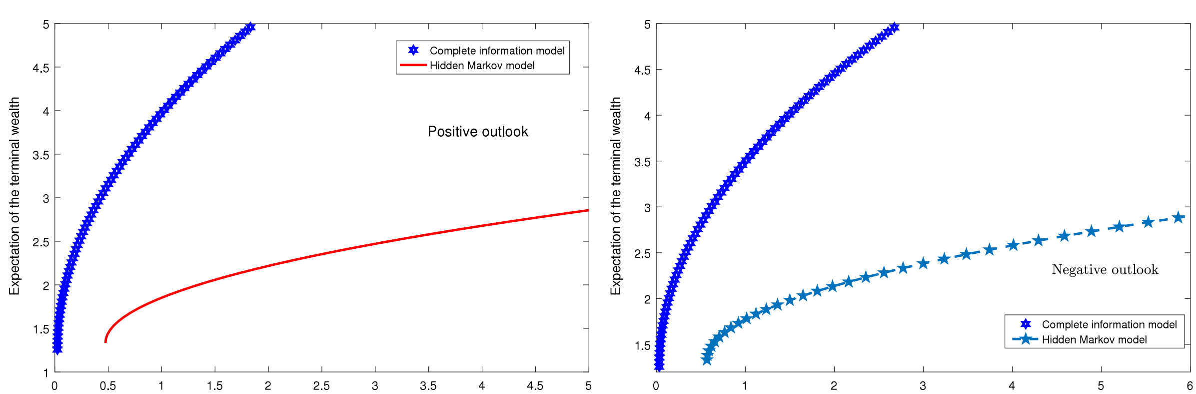

Fig. 2. Efficient frontiers of the hidden Markov model and the complete information model under the market observation of a positive and a negative outlook.

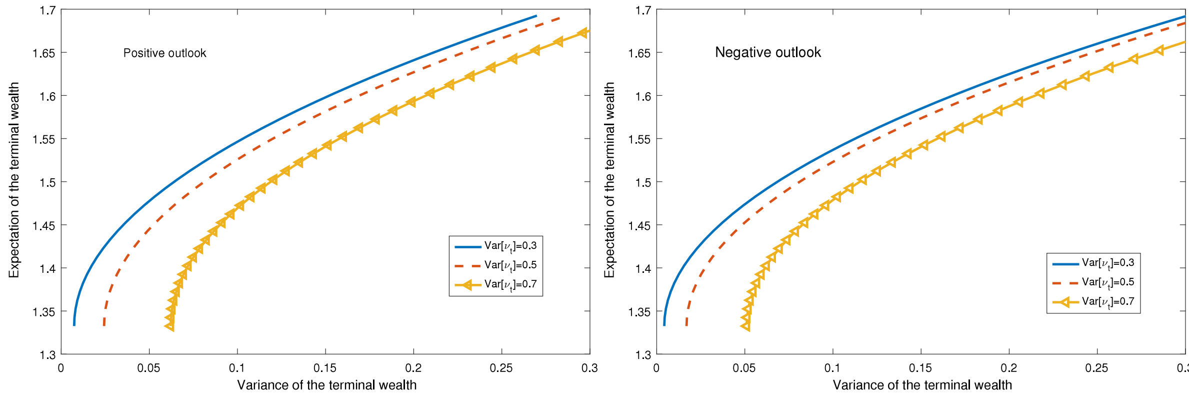

Fig. 3. The effects of Var$[\nu_t]$ on the efficient frontier of hidden Markov model.

Table 1
Decision-making process under a positive outlook observation process.

		$t=0$ $O_1=\mathrm{PO}$	$t=1$ $O_2=\mathrm{PO}$	$t=2$ $O_3=\mathrm{PO}$	$t=3$ $O_4=\mathrm{PO}$
Optimal investment amount on 3 assets under CIM	Asset 1	−9.9075	−8.3097	−6.0580	−4.4840
	Asset 2	12.0280	10.1025	7.9345	5.4006
	Asset 3	−6.1520	−5.1179	−4.0580	−2.7651
Optimal investment amount on 3 assets under HMM	Asset 1	−8.8430	−4.0292	−1.4228	−0.0816
	Asset 2	10.1456	4.5553	1.6000	0.0915
	Asset 3	−3.2097	−1.1699	−0.3757	−0.0205

Table 2
Decision-making process under a negative outlook observation process.

		$t=0$ $O_1=\mathrm{NO}$	$t=1$ $O_2=\mathrm{NO}$	$t=2$ $O_3=\mathrm{NO}$	$t=3$ $O_4=\mathrm{NO}$
Optimal investment amount on 3 assets under CIM	Asset 1	2.5250	2.3248	1.9387	1.5906
	Asset 2	1.6231	1.4439	1.2445	1.0215
	Asset 3	−1.3360	−1.1894	−1.0256	−0.8461
Optimal investment amount on 3 assets under HMM	Asset 1	−7.8462	−4.6422	−3.2242	−2.7453
	Asset 2	7.5026	4.0984	2.7667	2.3238
	Asset 3	2.4944	1.9710	1.6236	1.4846

Table 3
Sharpe ratios.

	$\overline{O}=(\mathrm{PO,PO,PO,PO})$	$\overline{O}=(\mathrm{NO,NO,NO,NO})$
CIM	5.4643	4.9967
HMM	3.0476	2.6697

In order to measure the investment value on the risky assets from time 0 to time T, we define the Sharpe ratio for the DC pension fund in discrete-time setting as follows

$$Sharpe[0,T]=\frac{\mathrm{E}_{0,j}(W_T)-\left(\prod_{t=0}^{T-1}r_t^0\right)W_0}{\sqrt{\mathrm{Var}_{0,j}(W_T)}} \tag{44}$$

which is similar to the definition in the static investment environment. Then, under the above mentioned observation processes $\overline{O}=(\mathrm{PO,PO,PO,PO})$ and $\overline{O}=(\mathrm{NO,NO,NO,NO})$, we obtain the Sharpe ratios for CIM and HMM respectively as follows (see Table 3).

As imagined, no matter for a positive or negative outlook observation process, the investment value on the risky assets for DC pension fund in case of imperfect information is smaller than that in case with complete observable information. In other words, the more market information the decision-maker receives, the more investment benefit she/he will obtain. On the other hand, comparing the Sharpe ratios of CIM and HMM, we find that the Sharpe ratio under a positive observation process is greater than that under a negative observation process. That is to say, the optimistic market performance implies a greater investment return comparing with the pessimistic marker observation.

7. Conclusion

Although most studies of the optimal DC pension fund management problem have been conducted under a setting of completely observable market information, this paper investigates the optimal investment strategy in a financial market with imperfect information for a multi-period mean–variance DC pension fund management problem. This setting is justified because decision-makers can receive only limited (incomplete) information about financial markets. Over the long investment period, more and more observed information updates the market observation process. Consequently, decision-makers make their investment decisions based on the information observed up to that moment. We formulate our imperfect information market by separating the financial market states into two parts, the observable market state and the unobservable market state, and we call it the hidden Markov model. The explicit expressions for the investment strategy and the efficient frontier are obtained under a mean–variance framework using sufficient statistics, the dynamic programming method, and the Lagrange dual theory. By adopting numerical analysis, the effects of the unobservable market information on the optimal investment strategy and efficient frontier are studied in detail. Our main results are as follows. (i) Due to the imperfection of the observed market information, decision-makers get less investment return with incompletely observable than completely observable information under the same level of investment risk; however, the more positive information is observed about the financial market, the higher the investment yield received. (ii) For the same level of investment risk, a greater variance in the salary growth rate increases the investment risk. (iii) The higher the contribution rate, the higher the investment yield, and this increase is more rapid under a positive than a negative outlook. (iv) In general, less money is invested in risky assets in a financial market with imperfect information than with complete information. However, in either case, the more positive signals decision-makers receive, the more they invest in risky assets.

Appendix A. Proof of Theorem 4.1

We prove this theorem using mathematical induction on t.

According to the Bellman equation (11) and the terminal condition $\mathrm{V}_T^*(\Phi^j(T), j, \mathrm{W}_T) = \mathrm{W}_T^2 - 2a\mathrm{W}_T$, we have the following formula at time point $t = T-1$ with the given observed market state $O_{T-1} = j$,

$$
\begin{aligned}
&\mathrm{V}_{T-1}^*(\Phi^j(T-1), j, \mathrm{W}_{T-1}) \\
&= \min_{\pi_{T-1}(j)} \mathrm{E}\Big[\mathrm{V}_T^*(\Phi^{O_T}(T), O_T, \mathrm{W}_T)|\Phi^j(T-1), j, \mathrm{W}_{T-1}\Big] \\
&= \min_{\pi_{T-1}(j)} \mathrm{E}\Big[\mathrm{W}_T^2 - 2a\mathrm{W}_T|\Phi^j(T-1), j, \mathrm{W}_{T-1}\Big] \\
&= \min_{\pi_{T-1}(j)} \sum_{i=1}^n \varphi^j(T-1, i)\sum_{k=1}^m \theta_{T-1}(i,k)\mathrm{E}\Big[\mathrm{W}_T^2 - 2a\mathrm{W}_T\Big] \\
&= \min_{\pi_{T-1}(j)} \sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}\Big\{ \big[r_{T-1}^0(i,j)\mathrm{W}_{T-1} + r_{T-1}^0(i,j)c_{T-1}(j)s_{T-1} \\
&\qquad + \mathrm{P}_{T-1}'(i,j)\pi_{T-1}(j)\big]^2 \\
&\qquad - 2a\big[r_{T-1}^0(i,j)\mathrm{W}_{T-1} + r_{T-1}^0(i,j)c_{T-1}(j)s_{T-1} \\
&\qquad + \mathrm{P}_{T-1}'(i,j)\pi_{T-1}(j)\big]\Big\} \\
&= \min_{\pi_{T-1}(j)} \sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}\Big\{\Big[(r_{T-1}^0(i,j))^2\mathrm{W}_{T-1}^2 + c_{T-1}(j)^2 \\
&\qquad \times (r_{T-1}^0(i,j))^2 s_{T-1}^2 \\
&\qquad + \pi_{T-1}'(j)[\mathrm{P}_{T-1}(i,j)\mathrm{P}_{T-1}'(i,j)]\pi_{T-1}(j) + 2c_{T-1}(j) \\
&\qquad \times [r_{T-1}^0(i,j)]^2\mathrm{W}_{T-1}s_{T-1} \\
&\qquad + 2c_{T-1}(j)[r_{T-1}^0(i,j)\mathrm{P}_{T-1}'(i,j)]s_{T-1}\pi_{T-1}(j) \\
&\qquad + 2[r_{T-1}^0(i,j)\mathrm{P}_{T-1}'(i,j)]\mathrm{W}_{T-1}\pi_{T-1}(j)\Big] \\
&\qquad - 2ar_{T-1}^0(i,j)\mathrm{W}_{T-1} - 2a[c_{T-1}(j)r_{T-1}^0(i,j)]s_{T-1} \\
&\qquad - 2a\mathrm{P}_{T-1}'(i,j)\pi_{T-1}(j)\Big\} \\
&= \min_{\pi_{T-1}(j)} \sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)]^2(\mathrm{W}_{T-1})^2 \\
&\qquad + \left[(c_{T-1}(j))^2 \sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)]^2\right] s_{T-1}^2 \\
&\qquad + \pi_{T-1}'(j)\left[\sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}[\mathrm{P}_{T-1}(i,j)\mathrm{P}_{T-1}'(i,j)]\right]\pi_{T-1}(j) \\
&\qquad + 2\mathrm{W}_{T-1}\left[\sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)\mathrm{P}_{T-1}'(i,j)]\right]\pi_{T-1}(j) \\
&\qquad + 2s_{T-1}\left[c_{T-1}(j)\sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)\mathrm{P}_{T-1}'(i,j)]\right] \\
&\qquad \times \pi_{T-1}(j) \\
&\qquad + 2\mathrm{W}_{T-1}\left[c_{T-1}(j)\sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)]^2\right] s_{T-1} \\
&\qquad - 2a\mathrm{W}_{T-1}\left[\sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)]\right] \\
&\qquad - 2as_{T-1}\left[c_{T-1}(j)\sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)]\right] \\
&\qquad - 2a\left[\sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}[\mathrm{P}_{T-1}'(i,j)]\right]\pi_{T-1}(j),
\end{aligned}
\tag{A.1}
$$

where the third equality is obtained using $\sum_{k=1}^m \theta_{T-1}(i,k) = 1$, and the fifth equality is obtained using $\sum_{i=1}^n \varphi_{T-1}^j(i) = 1$. Because for $j = 1, 2, \ldots, m$, $\mathrm{E}[\mathrm{P}_{T-1}(i,j)\mathrm{P}_{T-1}'(i,j)]$ is positive definite, $\varphi_{T-1}^j(i) \geq 0$ and $\sum_{i=1}^n \varphi_{T-1}^j(i) = 1$, then

$$
\sigma_{T-1}(j) = \sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}[\mathrm{P}_{T-1}(i,j)\mathrm{P}_{T-1}'(i,j)]
$$

is positive definite for all $j = 1, 2, \ldots, m$. Therefore, by the first order condition about $\pi_{T-1}(j)$, we obtain the optimal decision

$$
\begin{aligned}
\pi_{T-1}^*(j) &= a\sigma_{T-1}^{-1}(j)h_{T-1}(j) - \sigma_{T-1}^{-1}(j) \\
&\quad \times \Big[\eta_{T-1}(j)\mathrm{W}_{T-1} - u_{T-1}(j)s_{T-1}\Big],
\end{aligned}
\tag{A.2}
$$

where

$$
h_{T-1}(j) = \sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}[\mathrm{P}_{T-1}(i,j)],
$$

$$
\eta_{T-1}(j) = \sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)\mathrm{P}_{T-1}(i,j)],
$$

$$
u_{T-1}(j) = c_{T-1}(j)\sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)\mathrm{P}_{T-1}(i,j)].
$$

Substituting (A.2) back into (A.1) yields

$$
\begin{aligned}
&\mathrm{V}_{T-1}^*(\Phi^j(T-1), j, \mathrm{W}_{T-1}) \\
&\quad = -a^2 B_{T-1}(j) - 2a\Big[D_{T-1}^{(1)}(j)\mathrm{W}_{T-1} + D_{T-1}^{(2)}(j)s_{T-1}\Big] \\
&\qquad + A_{T-1}^{(1)}(j)(\mathrm{W}_{T-1})^2 + 2A_{T-1}^{(2)}(j)\mathrm{W}_{T-1}s_{T-1} + A_{T-1}^{(3)}(j)(s_{T-1})^2
\end{aligned}
$$

where

$$
B_{T-1}(j) = h_{T-1}'(j)\sigma_{T-1}^{-1}(j)h_{T-1}(j),
$$

$$
D_{T-1}^{(1)}(j) = \sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)] - h_{T-1}'(j)\sigma_{T-1}^{-1}(j)\eta_{T-1}(j),
$$

$$
\begin{aligned}
D_{T-1}^{(2)}(j) = c_{T-1}(j)\Bigg[&\sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)] \\
&- h_{T-1}'(j)\sigma_{T-1}^{-1}(j)\eta_{T-1}(j)\Bigg],
\end{aligned}
$$

$$
A_{T-1}^{(1)}(j) = \sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)]^2 - \eta_{T-1}'(j)\sigma_{T-1}^{-1}(j)\eta_{T-1}(j),
$$

$$
\begin{aligned}
A_{T-1}^{(2)}(j) = 2c_{T-1}(j)\Bigg[&\sum_{i=1}^n \varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)]^2 \\
&- \eta_{T-1}'(j)\sigma_{T-1}^{-1}(j)\eta_{T-1}(j)\Bigg],
\end{aligned}
$$

$$A^{(3)}_{T-1}(j) = (c_{T-1}(j))^2 \left[\sum_{i=1}^{n} \varphi^j_{T-1}(i) \mathrm{E}[r^0_{T-1}(i,j)]^2 - \eta'_{T-1}(j)\sigma^{-1}_{T-1}(j)\eta_{T-1}(j) \right].$$

Generally, we suppose that (10) holds true for $t+1$, i.e.

$$\begin{aligned} \mathrm{V}^*_{t+1}(\Phi^j(t+1), j, \mathrm{W}_{t+1}) &= -a^2 B_{t+1}(j) - 2a\Big[D^{(1)}_{t+1}(j)\mathrm{W}_{t+1} + D^{(2)}_{t+1}(j)s_{t+1}\Big] \\ &\quad + A^{(1)}_{t+1}(j)(\mathrm{W}_{t+1})^2 + 2A^{(2)}_{t+1}(j)\mathrm{W}_{t+1}s_{t+1} + A^{(3)}_{t+1}(j)(s_{t+1})^2. \end{aligned} \tag{A.3}$$

According to the Bellman equation (11), for t we have

$$\begin{aligned} &\mathrm{V}^*_t(\Phi^j(t), j, \mathrm{W}_t) \\ &= \min_{\pi_t(j)} \sum_{i=1}^{n} \varphi^j_t(i) \sum_{k=1}^{m} \theta_t(i,k) \mathrm{E}\Big[\mathrm{V}^*_{t+1}(\Phi^k(t+1), k, \mathrm{W}_{t+1})\Big] \\ &= \min_{\pi_t(j)} \sum_{i=1}^{n} \varphi^j_t(i) \sum_{k=1}^{m} \theta_t(i,k) \mathrm{E}\Big\{ -a^2 B_{t+1}(k) \\ &\qquad - 2a\Big[D^{(1)}_{t+1}(k)\mathrm{W}_{t+1} + D^{(2)}_{t+1}(k)s_{t+1}\Big] \\ &\qquad + A^{(1)}_{t+1}(k)(\mathrm{W}_{t+1})^2 + 2A^{(2)}_{t+1}(k)\mathrm{W}_{t+1}s_{t+1} + A^{(3)}_{t+1}(k)(s_{t+1})^2 \Big\} \\ &= \min_{\pi_t(j)} \; -a^2 \left[\sum_{i=1}^{n} \varphi^j_t(i) \sum_{k=1}^{m} \theta_t(i,k) B_{t+1}(k) \right] \\ &\qquad - 2a \sum_{i=1}^{n} \left\{ \varphi^j_t(i) \sum_{k=1}^{m} \theta_t(i,k) \mathrm{E}\Big\{ D^{(1)}_{t+1}(k) \left[r^0_t(i,j) \right.\right. \\ &\qquad \left.\left. \times\ (\mathrm{W}_t + c_t(j)s_t) + \mathrm{P}'_t(i,j)\pi_t(j) \right] \Big\} \right\} \\ &\qquad - 2a \sum_{i=1}^{n} \left\{ \varphi^j_t(i) \sum_{k=1}^{m} \theta_t(i,k) \mathrm{E}\Big[D^{(2)}_{t+1}(k)\nu_t(i,j)s_t \Big] \right\} \\ &\qquad + \sum_{i=1}^{n} \left\{ \varphi^j_t(i) \sum_{k=1}^{m} \theta_t(i,k) \mathrm{E}\left\{ A^{(1)}_{t+1}(k) \left[r^0_t(i,j) \right.\right.\right. \\ &\qquad \left. \times (\mathrm{W}_t + c_t(j)s_t) + \mathrm{P}'_t(i,j)\pi_t(j) \right]^2 \\ &\qquad + 2\nu_t(i,j)A^{(2)}_{t+1}(k) \left[r^0_t(i,j)(\mathrm{W}_t + c_t(j)s_t) + \mathrm{P}'_t(i,j)\pi_t(j) \right] s_t \\ &\qquad \left.\left. + (\nu_t(i,j))^2 A^{(3)}_{t+1}(k) s_t^2 \right\} \right\} \\ &= \min_{\pi_t(j)} \; -a^2 \sum_{i=1}^{n} \left[\varphi^j_t(i) \sum_{k=1}^{m} \theta_t(i,k) B_{t+1}(k) \right] \\ &\qquad - 2a \left[\sum_{i=1}^{n} \left[\varphi^j_t(i) \mathrm{E}[r^0_t(i,j)] \sum_{k=1}^{m} \theta_t(i,k) D^{(1)}_{t+1}(k) \right] \right] \mathrm{W}_t \\ &\qquad - 2a \left\{ c_t(j) \sum_{i=1}^{n} \left[\varphi^j_t(i) \mathrm{E}[r^0_t(i,j)] \sum_{k=1}^{m} \theta_t(i,k) D^{(1)}_{t+1}(k) \right] \right. \\ &\qquad \left. + \sum_{i=1}^{n} \left[\varphi^j_t(i) \mathrm{E}[\nu_t(i,j)] \sum_{k=1}^{m} \theta_t(i,k) D^{(2)}_{t+1}(k) \right] \right\} s_t \\ &\qquad - 2a \left[\sum_{i=1}^{n} \left[\varphi^j_t(i) \mathrm{E}[\mathrm{P}'_t(i,j)] \sum_{k=1}^{m} \theta_t(i,k) D^{(1)}_{t+1}(k) \right] \right] \pi_t(j) \\ &\qquad + \left[\sum_{i=1}^{n} \left[\varphi^j_t(i) \mathrm{E}[r^0_t(i,j)]^2 \sum_{k=1}^{m} \theta_t(i,k) A^{(1)}_{t+1}(k) \right] \right] (\mathrm{W}_t)^2 \\ &\qquad + \left\{ (c_t(j))^2 \sum_{i=1}^{n} \left[\varphi^j_t(i) \mathrm{E}[r^0_t(i,j)]^2 \sum_{k=1}^{m} \theta_t(i,k) A^{(1)}_{t+1}(k) \right] \right. \\ &\qquad + 2c_t(j) \sum_{i=1}^{n} \left[\varphi^j_t(i) \mathrm{E}[r^0_t(i,j)] \mathrm{E}[\nu_t(i,j)] \sum_{k=1}^{m} \theta_t(i,k) A^{(2)}_{t+1}(k) \right] \\ &\qquad \left. + \sum_{i=1}^{n} \left[\varphi^j_t(i) \mathrm{E}[\nu_t(i,j)]^2 \sum_{k=1}^{m} \theta_t(i,k) A^{(3)}_{t+1}(k) \right] \right\} (s_t)^2 \\ &\qquad + \pi'_t(j) \left[\sum_{i=1}^{n} \left[\varphi^j_t(i) \mathrm{E}[\mathrm{P}_t(i,j)\mathrm{P}'_t(i,j)] \sum_{k=1}^{m} \theta_t(i,k) A^{(1)}_{t+1}(k) \right] \right] \\ &\qquad \times \pi_t(j) \\ &\qquad + 2 \left[\sum_{i=1}^{n} \left[\varphi^j_t(i) \mathrm{E}[r^0_t(i,j)\mathrm{P}'_t(i,j)] \sum_{k=1}^{m} \theta_t(i,k) A^{(1)}_{t+1}(k) \right] \right] \\ &\qquad \times \mathrm{W}_t \pi_t(j) \\ &\qquad + 2 \left\{ c_t(j) \sum_{i=1}^{n} \left[\varphi^j_t(i) \mathrm{E}[r^0_t(i,j)\mathrm{P}'_t(i,j)] \sum_{k=1}^{m} \theta_t(i,k) A^{(1)}_{t+1}(k) \right] \right. \\ &\qquad \left. + \sum_{i=1}^{n} \left[\varphi^j_t(i) \mathrm{E}[\nu_t(i,j)] \mathrm{E}[\mathrm{P}'_t(i,j)] \sum_{k=1}^{m} \theta_t(i,k) A^{(2)}_{t+1}(k) \right] \right\} \\ &\qquad \times s_t \pi_t(j) \\ &\qquad + 2 \left[\sum_{i=1}^{n} \left[\varphi^j_t(i) \mathrm{E}[r^0_t(i,j)] \mathrm{E}[\nu_t(i,j)] \sum_{k=1}^{m} \theta_t(i,k) A^{(2)}_{t+1}(k) \right] \right] \\ &\qquad \times \mathrm{W}_t s_t. \end{aligned} \tag{A.4}$$

Notice that $A^{(1)}_{t+1}(k) > 0$ and $\mathrm{E}[P_t(i,j)P'_t(i,j)]$ is positive definite, therefore

$$\sigma_t(j) = \sum_{i=1}^{n} \varphi^j_t(i) \mathrm{E}[\mathrm{P}_t(i,j)\mathrm{P}'_t(i,j)] \sum_{k=1}^{m} \theta_t(i,k) A^{(1)}_{t+1}(k)$$

is positive definite. The optimal solution can now be derived using the first order condition about $\pi_t(j)$ given as

$$\pi^*_t(j) = a\sigma_t^{-1}(j)h_t(j) - \sigma_t^{-1}(j)\left[\eta_t(j)\mathrm{W}_t + u_t(j)s_t\right] \tag{A.5}$$

where

$$h_t(j) = \sum_{i=1}^{n} \left[\varphi^j_t(i) \mathrm{E}[\mathrm{P}_t(i,j)] \sum_{k=1}^{m} \theta_t(i,k) D^{(1)}_{t+1}(k) \right], \tag{A.6}$$

$$\eta_t(j) = \sum_{i=1}^{n} \left[\varphi^j_t(i) \mathrm{E}[r^0_t(i,j)\mathrm{P}_t(i,j)] \sum_{k=1}^{m} \theta_t(i,k) A^{(1)}_{t+1}(k) \right], \tag{A.7}$$

$$u_t(j) = c_t(j) \sum_{i=1}^{n} \left[\varphi^j_t(i) \mathrm{E}[r^0_t(i,j)\mathrm{P}_t(i,j)] \sum_{k=1}^{m} \theta_t(i,k) A^{(1)}_{t+1}(k) + \sum_{i=1}^{n} \varphi^j_t(i) \mathrm{E}[\nu_t(i,j)] \mathrm{E}[P_t(i,j)] \sum_{k=1}^{m} \theta_t(i,k) A^{(2)}_{t+1}(k) \right]. \tag{A.8}$$

By substituting $\pi^*_t(j)$ back into (A.4), the expression of the optimal value function $\mathrm{V}^*_t(\Phi^j(t), j, \mathrm{W}_t)$ can be analytically given by

$$\begin{aligned} \mathrm{V}^*_t(\Phi^j(t), j, \mathrm{W}_t) &= -a^2 B(j) - 2a\Big[D^{(1)}_t(j)\mathrm{W}_t + D^{(2)}_t(j)s_t\Big] \\ &\quad + A^{(1)}_t(j)(\mathrm{W}_t)^2 + 2A^{(2)}_t(j)\mathrm{W}_t s_t + A^{(3)}_t(j)(s_t)^2 \end{aligned}$$

where

$$\mathrm{B}_t(j) = h'_t(j)\sigma_t^{-1}(j)h_t(j) + \sum_{i=1}^{n} \left[\varphi^j_t(i) \sum_{k=1}^{m} \theta_t(i,k) \mathrm{B}_{t+1}(k) \right], \tag{A.9}$$

$$D^{(1)}_t(j) = \sum_{i=1}^{n} \left[\varphi^j_t(i) \mathrm{E}[(r^0_t(i,j))^2] \sum_{k=1}^{m} \theta_t(i,k) \mathrm{D}^{(1)}_{t+1}(k) \right]$$

$$
-h_t'(j)\sigma_t^{-1}(j)\eta_t(j), \tag{A.10}
$$

$$
\begin{aligned}
D_t^{(2)}(j) = {} & c_t(j)\left[\sum_{i=1}^{n}\varphi_t^j(i)\mathrm{E}[r_t^0(i,j)]\sum_{k=1}^{m}\theta_t(i,k)\mathrm{D}_{t+1}^{(1)}(k)\right] \\
& + \sum_{i=1}^{n}\left[\varphi_t^j(i)\mathrm{E}[\nu_t(i,j)]\sum_{k=1}^{m}\theta_t(i,k)\mathrm{D}_{t+1}^{(2)}(k)\right] \\
& - h_t'(j)\sigma_t^{-1}(j)u_t(j),
\end{aligned} \tag{A.11}
$$

$$
\begin{aligned}
A_t^{(1)}(j) = {} & \sum_{i=1}^{n}\left[\varphi_t^j(i)\mathrm{E}[(r_t^0(i,j))^2]\sum_{k=1}^{m}\theta_t(i,k)A_{t+1}^{(1)}(k)\right] \\
& - \eta_t'(j)\sigma_t^{-1}(j)\eta_t(j),
\end{aligned} \tag{A.12}
$$

$$
\begin{aligned}
A_t^{(2)}(j) = {} & c_t(j)\left[\sum_{i=1}^{n}\left[\varphi_t^j(i)\mathrm{E}[(r_t^0(i,j))^2]\sum_{k=1}^{m}\theta_t(i,k)A_{t+1}^{(1)}(k)\right]\right] \\
& + \sum_{i=1}^{n}\left[\varphi_t^j(i)\mathrm{E}[r_t^0(i,j)]\mathrm{E}[\nu_t(i,j)]\sum_{k=1}^{m}\theta_t(i,k)\mathrm{A}_{t+1}^{(2)}(k)\right] \\
& - \eta_t'(j)\sigma_t^{-1}(j)u_t(j),
\end{aligned} \tag{A.13}
$$

$$
\begin{aligned}
A_t^{(3)}(j) = {} & (c_t(j))^2\left[\sum_{i=1}^{n}\left[\varphi_t^j(i)\mathrm{E}[(r_t^0(i,j))^2]\sum_{k=1}^{m}\theta_t(i,k)A_{t+1}^{(1)}(k)\right]\right] \\
& + 2c_t(j)\left[\sum_{i=1}^{n}\left[\varphi_t^j(i)\mathrm{E}[r_t^0(i,j)]\mathrm{E}[\nu_t(i,j)]\right.\right. \\
& \quad \left.\left. \times \sum_{k=1}^{m}\theta_t(i,k)\mathrm{A}_{t+1}^{(2)}(k)\right]\right] \\
& + \sum_{i=1}^{n}\left[\varphi_t^j(i)\mathrm{E}[(\nu_t(i,j))^2]\sum_{k=1}^{m}\theta_t(i,k)\mathrm{A}_{t+1}^{(3)}(k)\right] \\
& - u_t'(j)\sigma_t^{-1}(j)u_t(j).
\end{aligned} \tag{A.14}
$$

For $t = 0, 1, \ldots, T-1$, and $B_T(j) = D_T^{(3)}(j) = A_T^{(2)}(j) = A_T^{(3)}(j) = 0$, $D_T^{(1)}(j) = A_T^{(1)}(j) = 1$.

This means that (12) and (13) hold true for t. Then, by mathematical induction, we complete the proof of the theorem.

Appendix B. Proof of Lemma 4.1

Because, at time t, for any given unobservable market state $U_t = i$ and observable market state $O_t = j$, $\mathrm{E}[\mathbf{r}_t(i,j)\mathbf{r}_t'(i,j)] = \mathrm{Cov}[\mathbf{r}_t(i,j)] + \mathrm{E}[\mathbf{r}_t(i,j)]\mathrm{E}[\mathbf{r}_t'(i,j)]$ is positive definite for all time periods, where $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$, we have

$$
\begin{aligned}
& \mathrm{E}[\mathbf{r}_t(i,j)\mathbf{r}_t'(i,j)] \\
& = \begin{bmatrix}
\mathrm{E}[r_t^0(i,j)]^2 & \mathrm{E}[r_t^0(i,j)r_t^1(i,j)] & \cdots & \mathrm{E}[r_t^0(i,j)r_t^L(i,j)] \\
\mathrm{E}[r_t^1(i,j)r_t^0(i,j)] & \mathrm{E}[r_t^1(i,j)]^2 & \cdots & \mathrm{E}[r_t^1(i,j)r_t^L(i,j)] \\
\cdots & \cdots & \cdots & \cdots \\
\mathrm{E}[r_t^L(i,j)r_t^0(i,j)] & \mathrm{E}[r_t^L(i,j)r_t^1(i,j)] & \cdots & \mathrm{E}[r_t^L(i,j)]^2
\end{bmatrix} \\
& > 0,
\end{aligned} \tag{B.1}
$$

for $t = 0, 1, \ldots, T-1$. Then, following Eq. (B.1), we have

$$
\begin{aligned}
& \begin{bmatrix}
\mathrm{E}[r_t^0(i,j)]^2 & \mathrm{E}[r_t^0(i,j)\mathrm{P}_t'(i,j)] \\
\mathrm{E}[r_t^0(i,j)\mathrm{P}_t(i,j)] & \mathrm{E}[\mathrm{P}_t(i,j)\mathrm{P}_t'(i,j)]
\end{bmatrix} \\
& = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
-1 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
-1 & 0 & \cdots & 1
\end{bmatrix} \mathrm{E}[\mathbf{r}_t(i,j)\mathbf{r}_t'(i,j)] \\
& \quad \times \begin{bmatrix}
1 & -1 & \cdots & -1 \\
0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1
\end{bmatrix} > 0.
\end{aligned} \tag{B.2}
$$

According to (B.2), we obtain

$$
\begin{aligned}
& \mathrm{E}[\mathrm{P}_t(i,j)\mathrm{P}_t'(i,j)] > 0, \quad \forall t = 0, 1, \ldots, T-1, \\
& \quad i = 1, 2, \ldots, n \quad \text{and} \quad j = 1, 2, \ldots, m.
\end{aligned} \tag{B.3}
$$

Because $\varphi_{T-1}^j(i) > 0$ for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$, we have

$$
\sum_{i=1}^{n}\varphi_{T-1}^j(i)\mathrm{E}[\mathrm{P}_{T-1}(i,j)\mathrm{P}_{T-1}'(i,j)] > 0 \tag{B.4}
$$

and

$$
\begin{aligned}
& \sum_{i=1}^{n}\varphi_{T-1}^j(i)\begin{bmatrix}
\mathrm{E}[r_{T-1}^0(i,j)]^2 & \mathrm{E}[r_{T-1}^0(i,j)\mathrm{P}_{T-1}'(i,j)] \\
\mathrm{E}[r_{T-1}^0(i,j)\mathrm{P}_t(i,j)] & \mathrm{E}[\mathrm{P}_{T-1}(i,j)\mathrm{P}_{T-1}'(i,j)]
\end{bmatrix} \\
& = \begin{bmatrix}
\sum_{i=1}^{n}\varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)]^2 & \sum_{i=1}^{n}\varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)\mathrm{P}_{T-1}'(i,j)] \\
\sum_{i=1}^{n}\varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)\mathrm{P}_{T-1}(i,j)] & \sum_{i=1}^{n}\varphi_{T-1}^j(i)\mathrm{E}[\mathrm{P}_{T-1}(i,j)\mathrm{P}_{T-1}'(i,j)]
\end{bmatrix} \\
& > 0,
\end{aligned} \tag{B.5}
$$

which implies that

$$
\begin{aligned}
& \sum_{i=1}^{n}\varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)]^2 - \left[\sum_{i=1}^{n}\varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)\mathrm{P}_{T-1}'(i,j)]\right] \cdot \\
& \left[\sum_{i=1}^{n}\varphi_{T-1}^j(i)\mathrm{E}[\mathrm{P}_{T-1}(i,j)\mathrm{P}_{T-1}'(i,j)]\right]^{-1} \\
& \cdot \left[\sum_{i=1}^{n}\varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)\mathrm{P}_{T-1}(i,j)]\right] \\
& > 0,
\end{aligned} \tag{B.6}
$$

i.e.

$$
A_{T-1}^{(1)}(j) = \sum_{i=1}^{n}\varphi_{T-1}^j(i)\mathrm{E}[r_{T-1}^0(i,j)]^2 - \eta_{T-1}'(j)\sigma_{T-1}^{-1}(j)\eta_{T-1}(j) > 0.
$$

Suppose that at time $t+1$ we have $A_{t+1}^{(1)}(j) > 0$. Then

$$
\tau^j(t,i) = \varphi_t^j(i)\sum_{k=1}^{m}\theta_t(i,k)A_{t+1}^{(1)}(k) > 0,
$$

and

$$
\sum_{i=1}^{n}\tau^j(t,i)\mathrm{E}[\mathrm{P}_t(i,j)\mathrm{P}_t'(i,j)] > 0.
$$

Furthermore,

$$
\begin{aligned}
& \sum_{i=1}^{n}\tau^j(t,i)\begin{bmatrix}
\mathrm{E}[r_t^0(i,j)]^2 & \mathrm{E}[r_t^0(i,j)\mathrm{P}_t'(i,j)] \\
\mathrm{E}[r_t^0(i,j)\mathrm{P}_t(i,j)] & \mathrm{E}[\mathrm{P}_t(i,j)\mathrm{P}_t'(i,j)]
\end{bmatrix} \\
& = \begin{bmatrix}
\sum_{i=1}^{n}\tau^j(t,i)\mathrm{E}[r_t^0(i,j)]^2 & \sum_{i=1}^{n}\tau^j(t,i)\mathrm{E}[r_t^0(i,j)\mathrm{P}_t'(i,j)] \\
\sum_{i=1}^{n}\tau^j(t,i)\mathrm{E}[r_t^0(i,j)\mathrm{P}_t(i,j)] & \sum_{i=1}^{n}\tau^j(t,i)\mathrm{E}[\mathrm{P}_t(i,j)\mathrm{P}_t'(i,j)]
\end{bmatrix} \\
& > 0,
\end{aligned} \tag{B.7}
$$

which implies that

$$
\begin{aligned}
& \sum_{i=1}^{n}\tau^j(t,i)\mathrm{E}[r_t^0(i,j)]^2 - \left[\sum_{i=1}^{n}\tau^j(t,i)\mathrm{E}[r_t^0(i,j)\mathrm{P}_t'(i,j)]\right] \cdot \\
& \left[\sum_{i=1}^{n}\tau^j(t,i)\mathrm{E}[\mathrm{P}_t(i,j)\mathrm{P}_t'(i,j)]\right]^{-1} \cdot \left[\sum_{i=1}^{n}\tau^j(t,i)\mathrm{E}[r_t^0(i,j)\mathrm{P}_t(i,j)]\right] > 0.
\end{aligned}
$$

That is

$$A_t^{(1)}(j) = \sum_{i=1}^{n} \varphi_t^j(i) \mathrm{E}[r_t^0(i,j)]^2 \sum_{k=1}^{m} \theta_t(i,k) A_{t+1}^{(1)}(k) - \eta_t'(j)\sigma_t^{-1}(j)\eta_t(j) > 0$$

for $j = 1, 2, \ldots, m$. This means that $A_t^1(j) > 0$ holds true for t. By the principle of mathematical induction, we can conclude that $A_t^1(j) > 0$ holds for $t = 0, 1, \ldots, T-1$ and $j = 1, 2, \ldots, m$.

Appendix C. Proof of Lemma 4.2

First, we need some basic knowledge about the positive definite matrix, which we list here.

N1. In this whole paper, if a matrix M is positive definite, we denote it by $\mathrm{M} > \mathbf{0}$.

N2. For positive definite matrices, we define the partial order relation between the matrices as follows. If M and N are two positive definite matrices, i.e. $\mathrm{M} > \mathbf{0}$ and $\mathrm{N} > \mathbf{0}$, and they satisfy $\mathrm{M} - \mathrm{N} > \mathbf{0}$, then $\mathrm{M} > \mathrm{N} > \mathbf{0}$.

N3. If M and N are two positive definite matrices which satisfy $\mathrm{M} > \mathrm{N} > \mathbf{0}$, then $\mathrm{N}^{-1} > \mathrm{M}^{-1} > \mathbf{0}$.

N4. If M and N are two positive definite matrices which satisfy $\mathrm{MN} = \mathrm{NM}$, then $\mathrm{MN} > \mathbf{0}$.

Proof. As $\xi_i = [\xi_i^1, \xi_i^2, \ldots, \xi_i^n]'$ is non-degenerate and Σ_i is positive definite, then $\mathrm{E}[\xi_i\xi_i'] = \Sigma_i + \mathrm{E}[\xi_i]\mathrm{E}[\xi_i']$ is positive definite for any $i = 1, 2, \ldots, n$. Therefore,

$$\begin{aligned}
\sum_{i=1}^{n} \tau_i \mathrm{E}[\xi_i\xi_i'] &= \sum_{i=1}^{n} \tau_i \Sigma_i + \sum_{i=1}^{m} \tau_i \mathrm{E}[\xi_i]\mathrm{E}[\xi_i'] \\
&= \sum_{i=1}^{n} \tau_i \Sigma_i + \left(\mathrm{E}[\sqrt{\tau_1}\xi_1], \mathrm{E}[\sqrt{\tau_2}\xi_2], \ldots, \mathrm{E}[\sqrt{\tau_n}\xi_n]\right) \\
&\quad \times \begin{pmatrix} \mathrm{E}[\sqrt{\tau_1}\xi_1'] \\ \mathrm{E}[\sqrt{\tau_2}\xi_2'] \\ \vdots \\ \mathrm{E}[\sqrt{\tau_n}\xi_n'] \end{pmatrix} \\
&= \Sigma + \mathrm{HH}'
\end{aligned} \tag{C.1}$$

is also positive definite, where $\Sigma = \sum_{i=1}^{n} \tau_i \Sigma_i$ and $\mathrm{H} = \left[\mathrm{E}[\sqrt{\tau_1}\xi_1], \mathrm{E}[\sqrt{\tau_2}\xi_2], \ldots, \mathrm{E}[\sqrt{\tau_n}\xi_n]\right]$. We can also get the inverse matrix of $\sum_{i=1}^{n} \tau_i \mathrm{E}[\xi_i\xi_i']$ as follows,

$$\left[\sum_{i=1}^{n} \tau_i \mathrm{E}[\xi_i\xi_i']\right]^{-1} = \Sigma^{-1} - \Sigma^{-1}\mathrm{H}(\mathrm{I}_n + \mathrm{H}'\Sigma^{-1}\mathrm{H})^{-1}\mathrm{H}'\Sigma^{-1}, \tag{C.2}$$

where I_n is an n-dimension identical matrix. Moreover, we have

$$\begin{aligned}
&\mathrm{H}'\left[\Sigma^{-1} - \Sigma^{-1}\mathrm{H}(\mathrm{I}_n + \mathrm{H}'\Sigma^{-1}\mathrm{H})^{-1}\mathrm{H}'\Sigma^{-1}\right]\mathrm{H} \\
&= \mathrm{H}'\Sigma^{-1}\mathrm{H} - \mathrm{H}'\Sigma^{-1}\mathrm{H}(\mathrm{I}_n + \mathrm{H}'\Sigma^{-1}\mathrm{H})^{-1}\mathrm{H}'\Sigma^{-1}\mathrm{H} \\
&= \mathrm{H}'\Sigma^{-1}\mathrm{H}(\mathrm{I}_n + \mathrm{H}'\Sigma^{-1}\mathrm{H})^{-1}\left[\mathrm{I}_n + \mathrm{H}'\Sigma^{-1}\mathrm{H} - \mathrm{H}'\Sigma^{-1}\mathrm{H}\right] \\
&= \mathrm{H}'\Sigma^{-1}\mathrm{H}(\mathrm{I}_n + \mathrm{H}'\Sigma^{-1}\mathrm{H})^{-1}.
\end{aligned} \tag{C.3}$$

Because $\Sigma_i > \mathbf{0}$ for every $j = 1, 2, \ldots, m$, then $\Sigma > \mathbf{0}$ and $(\mathrm{I}_n + \mathrm{H}'\Sigma^{-1}\mathrm{H}) > \mathrm{H}'\Sigma^{-1}\mathrm{H} > \mathbf{0}$. Further, $(\mathrm{H}'\Sigma^{-1}\mathrm{H})^{-1} > (\mathrm{I}_n + \mathrm{H}'\Sigma^{-1}\mathrm{H})^{-1} > \mathbf{0}$. Notice that

$$\begin{aligned}
&\mathrm{H}'\Sigma^{-1}\mathrm{H}\left[(\mathrm{H}'\Sigma^{-1}\mathrm{H})^{-1} - (\mathrm{I}_n + \mathrm{H}'\Sigma^{-1}\mathrm{H})^{-1}\right] \\
&= \left[(\mathrm{H}'\Sigma^{-1}\mathrm{H})^{-1} - (\mathrm{I}_n + \mathrm{H}'\Sigma^{-1}\mathrm{H})^{-1}\right]\mathrm{H}'\Sigma^{-1}\mathrm{H} \\
&= \mathrm{I}_n - \mathrm{H}'\Sigma^{-1}\mathrm{H}(\mathrm{I}_n + \mathrm{H}'\Sigma^{-1}\mathrm{H})^{-1},
\end{aligned} \tag{C.4}$$

so we obtain

$$\mathrm{I}_n - \mathrm{H}'\Sigma^{-1}\mathrm{H}(\mathrm{I}_n + \mathrm{H}'\Sigma^{-1}\mathrm{H})^{-1} > \mathbf{0}. \tag{C.5}$$

Also

$$(\sqrt{\tau_1}, \sqrt{\tau_2}, \ldots, \sqrt{\tau_n})\left[\mathrm{I}_n - \mathrm{H}'\Sigma^{-1}\mathrm{H}(\mathrm{I}_n + \mathrm{H}'\Sigma^{-1}\mathrm{H})^{-1}\right] \times \begin{pmatrix} \sqrt{\tau_1} \\ \sqrt{\tau_2} \\ \vdots \\ \sqrt{\tau_n} \end{pmatrix} > 0, \tag{C.6}$$

which means that

$$\begin{aligned}
&(\sqrt{\tau_1}, \sqrt{\tau_2}, \ldots, \sqrt{\tau_n})\mathrm{H}'\Sigma^{-1}\mathrm{H}(\mathrm{I}_n + \mathrm{H}'\Sigma^{-1}\mathrm{H})^{-1} \begin{pmatrix} \sqrt{\tau_1} \\ \sqrt{\tau_2} \\ \vdots \\ \sqrt{\tau_n} \end{pmatrix} \\
&< (\sqrt{\tau_1}, \sqrt{\tau_2}, \ldots, \sqrt{\tau_n})\mathrm{I}_n \begin{pmatrix} \sqrt{\tau_1} \\ \sqrt{\tau_2} \\ \vdots \\ \sqrt{\tau_n} \end{pmatrix} \\
&= \sum_{i=1}^{n} \tau_i = 1.
\end{aligned}$$

Finally,

$$\begin{aligned}
0 &< \left[\sum_{i=1}^{n} \tau_i \mathrm{E}[\xi_i]\right]' \left[\sum_{i=1}^{n} \tau_i \mathrm{E}[\xi_i\xi_i']\right]^{-1} \left[\sum_{i=1}^{n} \tau_i \mathrm{E}[\xi_i]\right] \\
&= (\sqrt{\tau_1}, \sqrt{\tau_2}, \ldots, \sqrt{\tau_n})\mathrm{H}'\Sigma^{-1}\mathrm{H}(\mathrm{I}_n + \mathrm{H}'\Sigma^{-1}\mathrm{H})^{-1} \\
&\quad \times \begin{pmatrix} \sqrt{\tau_1} \\ \sqrt{\tau_2} \\ \vdots \\ \sqrt{\tau_n} \end{pmatrix} < \sum_{i=1}^{n} \tau_i = 1.
\end{aligned}$$

And we complete the proof.

Appendix D. Proof of Lemma 4.3

Proof. Notice that

$$\begin{aligned}
B_t(j) = &\left[\sum_{i=1}^{n} \left(\varphi_t^j(i) \sum_{k=1}^{m} \theta_t(i,k)\,(1 - B_{t+1}(k))\right) \mathrm{E}[\mathrm{P}_t(i,j)]\right]' \\
&\cdot \left[\sum_{i=1}^{n} \varphi_t^j(i) \left(\varphi_t^j(i) \sum_{k=1}^{m} \theta_t(i,k)\,(1 - B_{t+1}(k))\right) \right. \\
&\left. \mathrm{E}[P_t(i,j)\mathrm{P}_t'(i,j)]\right] \\
&\cdot \left[\sum_{i=1}^{n} \varphi_t^j(i) \left(\varphi_t^j(i) \sum_{k=1}^{m} \theta_t(i,k)\,(1 - B_{t+1}(k))\right) \mathrm{E}[\mathrm{P}_t(i,j)]\right] \\
&+ \sum_{i=1}^{n} \varphi_t^j(i) \sum_{k=1}^{m} \theta_t(i,k) B_{t+1}(k).
\end{aligned} \tag{D.1}$$

So let

$$\varrho^j(t,i) = \varphi_t^j(i) \sum_{k=1}^{m} \theta_t(i,k)\,(1 - B_{t+1}(k)),$$

then (D.1) can be rewritten as

$$\begin{aligned} B_t(j) = & \left[\sum_{i=1}^{n} \varrho^j(t,i)\mathrm{E}[\mathrm{P}_t(i,j)]\right]' \left[\sum_{i=1}^{n} \varrho^j(t,i)\mathrm{E}[P_t(i,j)\mathrm{P}_t'(i,j)]\right] \\ & \times \left[\sum_{i=1}^{n} \varrho^j(t,i)\mathrm{E}[\mathrm{P}_t(i,j)]\right] \\ & + \sum_{i=1}^{n} \varphi_t^j(i) \sum_{k=1}^{m} \theta_t(i,k) B_{t+1}(k). \end{aligned} \tag{D.2}$$

According to (14), we have

$$\begin{aligned} 0 < B_t(j) = & \left[\sum_{i=1}^{n} \varrho^j(t,i)\mathrm{E}[\mathrm{P}_t(i,j)]\right]' \left[\sum_{i=1}^{n} \varrho^j(t,i)\mathrm{E}[P_t(i,j)\mathrm{P}_t'(i,j)]\right] \\ & \times \left[\sum_{i=1}^{n} \varrho^j(t,i)\mathrm{E}[\mathrm{P}_t(i,j)]\right] \\ & + \sum_{i=1}^{n} \varphi_t^j(i) \sum_{k=1}^{m} \theta_t(i,k) B_{t+1}(k) \\ < & \sum_{i=1}^{n} \varrho^j(t,i) + \sum_{i=1}^{n} \varphi_t^j(i) \sum_{k=1}^{m} \theta_t(i,k) B_{t+1}(k) \\ = & \sum_{i=1}^{n} \varphi_t^j(i) \sum_{k=1}^{m} \theta_t(i,k) \left(1 - B_{t+1}(k)\right) \\ & + \sum_{i=1}^{n} \varphi_t^j(i) \sum_{k=1}^{m} \theta_t(i,k) B_{t+1}(k) \\ = & \sum_{i=1}^{n} \varphi_t^j(i) \sum_{k=1}^{m} \theta_t(i,k) = 1. \end{aligned} \tag{D.3}$$

Therefore, we complete the proof.

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